Number Systems

Animation 1.1: Complex Plane
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1.1 Introduction

In the very beginning, human life was simple. An early ancient herdsman compared sheep (or cattle) of his herd with a pile of stones when the herd left for grazing and again on its return for missing animals. In the earliest systems probably the vertical strokes or bars such as I, II, III, IIII etc.. were used for the numbers 1, 2, 3, 4 etc. The symbol "IIIII" was used by many people including the ancient Egyptians for the number of fingers of one hand.

Around 5000 B.C, the Egyptians had a number system based on 10. The symbol ∩ for 10 and 𓍢 for 100 were used by them. A symbol was repeated as many times as it was needed. For example, the numbers 13 and 324 were symbolized as ∩III and 𓍢𓍢𓍢∩∩IIII respectively. The symbol 𓍢𓍢𓍢∩∩IIII was interpreted as 100 + 100 +100+10+10+1+1+1 +1. Different people invented their own symbols for numbers. But these systems of notations proved to be inadequate with advancement of societies and were discarded. Ultimately the set {1, 2, 3, 4, ...} with base 10 was adopted as the counting set (also called the set of natural numbers). The solution of the equation $x + 2 = 2$ was not possible in the set of natural numbers, So the natural number system was extended to the set of whole numbers. No number in the set of whole numbers W could satisfy the equation $x + 4 = 2$ or $x + a = b$, if $a > b$, and $a, b, \in W$. The negative integers –1, –2, –3, ... were introduced to form the set of integers Z = {0, ±1, ±2 ,...).

Again the equation of the type $2x = 3$ or $bx = a$ where $a,b,\in Z$ and $b \neq 0$ had no solution in the set Z, so the numbers of the form $\frac{a}{b}$ where $a,b,\in Z$ and $b \neq 0$, were invented to remove such difficulties. The set $Q = \{\frac{a}{b} \mid a,b,\in Z \wedge b \neq 0\}$ was named as the set of rational numbers. Still the solution of equations such as $x^2 = 2$ or $x^2 = a$ (where a is not a perfect square) was not possible in the set Q. So the irrational numbers of the type $\pm\sqrt{2}$ or $\pm\sqrt{a}$ where a is not a perfect square were introduced. This process of enlargement of the number system ultimately led to the set of real numbers $\Re = Q \cup Q'$ (Q' is the set of irrational numbers) which is used most frequently in everyday life.

1.2 Rational Numbers and Irrational Numbers

We know that a rational number is a number which can be put in the form $\frac{p}{q}$ where $p, q \in Z \wedge q \neq 0$. The numbers $\sqrt{16}$, 3.7, 4 etc., are rational numbers. $\sqrt{16}$ can be reduced to the form $\frac{p}{q}$ where $p, q \in Z$, and $q \neq 0$ because $\sqrt{16} = 4 = \frac{4}{1}$.

Irrational numbers are those numbers which cannot be put into the form $\frac{p}{q}$ where $p, q \in Z$ and $q \neq 0$. The numbers $\sqrt{2}, \sqrt{3}, \frac{7}{\sqrt{5}}, \sqrt{\frac{5}{16}}$ are irrational numbers.

1.2.1 Decimal Representation of Rational and Irrational Numbers

1) Terminating decimals: A decimal which has only a finite number of digits in its decimal part, is called a terminating decimal. Thus 202.04, 0.0000415, 100000.41237895 are examples of terminating decimals.

Since a terminating decimal can be converted into a common fraction, so every terminating decimal represents a **rational number**.

2) Recurring Decimals: This is another type of **rational numbers**. In general, a recurring or periodic decimal is a decimal in which one or more digits repeat indefinitely.

It will be shown (in the chapter on sequences and series) that a recurring decimal can be converted into a common fraction. So **every recurring decimal represents a rational number:**

A non-terminating, non-recurring decimal is a decimal which neither terminates nor it is recurring. It is not possible to convert such a decimal into a common fraction. Thus a **non-terminating, non-recurring decimal represents an irrational number.**

Example 1:

i) .25 $(=\frac{25}{100})$ is a rational number.

ii) .333...$(=\frac{1}{3})$ is a recurring decimal, it is a rational number.

iii) $2.\bar{3}(=2.333...)$ is a rational number.

iv) 0.142857142857... $(=\frac{1}{7})$ is a rational number.

v) 0.01001000100001 ... is a non-terminating, non-periodic decimal, so it is an irrational number.

vi) 214.121122111222 1111 2222 ... is also an irrational number.

vii) 1.4142135 ... is an irrational number.

viii) 7.3205080 ... is an irrational number.

ix) 1.709975947 ... is an irrational number.

x) 3.141592654... is an important irrational number called it π(Pi) which denotes the constant ratio of the circumference of any circle to the length of its diameter i.e.,

$$\pi = \frac{\text{circumference of any circle}}{\text{length of its diameter.}}$$

An approximate value of π is $\frac{22}{7}$,a better approximation is $\frac{355}{113}$ and a still better approximation is 3.14159. The value of π correct to 5 lac decimal places has been determined with the help of computer.

Example 2: Prove $\sqrt{2}$ is an irrational number.

Solution: Suppose, if possible, $\sqrt{2}$ is rational so that it can be written in the form *p/q* where *p,q* $\in$ Z and $q \neq 0$. Suppose further that p/q is in its lowest form.

Then $\sqrt{2}$ = *p/q*, $(q \neq 0)$

Squaring both sides we get;

$$2 = \frac{p^2}{q^2} \text{ or } p^2 = 2q^2 \qquad (1)$$

The R.H.S. of this equation has a factor 2. Its L.H.S. must have the same factor.

Now a prime number can be a factor of a square only if it occurs at least twice in the square. Therefore, p^2 should be of the form $4p'^2$

so that equation (1) takes the form:

$$4p'^2 = 2q^2 \qquad ....(2)$$

i.e., $$2p'^2 = q^2 \qquad ....(3)$$

In the last equation, 2 is a factor of the L.H.S. Therefore, q^2 should be of the form $4q'^2$ so that equation 3 takes the form

$$2p'^2 = 4q'^2 \quad \text{i.e., } p'^2 = 2q'^2 \qquad ....(4)$$

From equations (1) and (2),

$$p = 2p'$$

and from equations (3) and (4)

$$q = 2q'$$

$$\therefore \quad \frac{p}{q} = \frac{2p'}{2q'}$$

This contradicts the hypothesis that $\frac{p}{q}$ is in its lowest form. Hence $\sqrt{2}$ is irrational.

Example 3: Prove $\sqrt{3}$ is an irrational number.

Solution: Suppose, if possible $\sqrt{3}$ is rational so that it can be written in the form *p/q* when *p,q* $\in$ *Z* and $q \neq 0$. Suppose further that *p/q* is in its lowest form,

then $\sqrt{3}$ = *p/q* , $(q \neq 0)$

Squaring this equation we get;

$$3 = \frac{p^2}{q^2} \qquad \text{or } p^2 = 3q^2 \qquad ....(1)$$

The R.H.S. of this equation has a factor 3. Its L.H.S. must have the same factor.

Now a prime number can be a factor of a square only if it occurs at least twice in the square. Therefore, p^2 should be of the form $9p'^2$ so that equation (1) takes the form:

$9p'^2 = 3q^2$ (2)

i.e., $3p'^2 = q^2$ (3)

In the last equation, 3 is a factor of the L.H.S. Therefore, q^2 should be of the form $9q'^2$ so that equation (3) takes the form

$3p'^2 = 9q'^2$ i.e., $p'^2 = 3q'^2$ (4)

From equations (1) and (2),

$P = 3P'$

and from equations (3) and (4)

$q = 3q'$

$$\therefore \quad \frac{p}{q} = \frac{3p'}{3q'}$$

This contradicts the hypothesis that $\frac{p}{q}$ is in its lowest form.

Hence $\sqrt{3}$ is irrational.

Note: Using the same method we can prove the irrationality of $\sqrt{5}, \sqrt{7}, \ldots, \sqrt{n}$ where n is any prime number.

1.3 Properties of Real Numbers

We are already familiar with the set of real numbers and most of their properties. We now state them in a unified and systematic manner. Before stating them we give a preliminary definition.

Binary Operation: A binary operation may be defined as a function from A $\times$ A into A, but for the present discussion, the following definition would serve the purpose. A *binary operation* in a set A is a rule usually denoted by * that assigns to any pair of elements of A, taken in a definite order, another element of A.

Two important binary operations are addition and multiplication in the set of real numbers. Similarly, union and intersection are binary operations on sets which are subsets of the

same Universal set.

$\Re$ usually denotes the set of real numbers. We assume that two binary operations addition (+) and multiplication (. or x) are defined in $\Re$. Following are the properties or laws for real numbers.

1. Addition Laws: -

i) Closure Law of Addition

$\forall\ a, b \in \Re,\ a + b \in \Re$ ($\forall$ stands for "for all")

ii) Associative Law of Addition

$\forall\ a, b, c \in \Re,\ a + (b + c) = (a + b) + c$

iii) Additive Identity

$\forall\ a \in \Re,\ \exists\ 0 \in \Re$ such that $a + 0 = 0 + a = a$

($\exists$ stands for "there exists").

0(read as zero) is called the identity element of addition.

iv) Additive Inverse

$\forall\ a \in \Re,\ \exists\ (-a) \in \Re$ such that

$a + (-a) = 0 = (-a) + a$

v) Commutative Law for Addition

$\forall\ a, b \in \Re,\ a + b = b + a$

2. Multiplication Laws

vi) Closure I.aw of Multiplication

$\forall\ a, b \in \Re,\ a.b \in \Re$ (a,b is usually written as ab).

vii) Associative Law for Multiplication

$\forall\ a, b, c \in \Re,\ a(bc) = (ab)c$

viii) Multiplicative Identity

$\forall\ a \in \Re,\ \exists\ 1 \in \Re$ such that $a.1 = 1.a = a$

1 is called the multiplicative identity of real numbers.

ix) Multiplicative Inverse

$\forall\ a (\neq 0) \in \Re,\ \exists\ a^{-1} \in \Re$ such that $a.a^{-1} = a^{-1}.a = 1$ (a^{-1} is also written as $\frac{1}{a}$).

x) Commutative Law of multiplication

$\forall\ a, b \in \Re,\ ab = ba$

3. Multiplication – Addition Law

xi) $\forall$ ***a, b, c*** $\in \Re$,

$a(b+c) = ab + ac$ (Distrihutivity of multiplication over addition).

$(a+b)c = ac + bc$

In addition to the above properties $\Re$ possesses the following properties.

i) Order Properties (described below).

ii) Completeness axiom which will be explained in higher classes.

The above properties characterizes $\Re$ i.e., only $\Re$ possesses all these properties.

Before stating the order axioms we state the properties of equality of numbers.

4. Properties of Equality

Equality of numbers denoted by "=" possesses the following properties:-

i) Reflexive property $\forall\ a \in \Re,\ a = a$

ii) Symmetric Property $\forall\ a,b \in \Re,\ a = b \Rightarrow b = a$.

iii) Transitive Property $\forall\ a,b,c \in \Re,\ a = b \wedge b = c \Rightarrow a = c$

iv) Additive Property $\forall\ a,b,c \in \Re,\ a = b \Rightarrow a + c = b + c$

v) Multiplicative Property $\forall\ a,b,c \in \Re,\ a = b \Rightarrow ac = bc \wedge ca = cb$.

vi) Cancellation Property w.r.t. addition

$\forall\ a,b,c \in \Re,\ a + c = b + c \Rightarrow a = b$

vii) Cancellation Property w.r.t. Multiplication:

$\forall\ a,b,c \in \Re,\ ac = bc \Rightarrow a = b,\ c \neq 0$

5. Properties of Ineualities (Order properties)

1) Trichotomy Property $\forall\ a,b \in \Re$

either $a = b$ or $a > b$ or $a < b$

2) Transitive Property $\forall\ a,b,c \in \Re$

i) $a > b \wedge b > c \Rightarrow a > c$ ii) $a < b \wedge b < c \Rightarrow a < c$

3) Additive Property: $\forall\ a,b,c \in \Re$

a) i) $a > b \Rightarrow a + c > b + c$ b) i) $a > b \wedge c > d \Rightarrow a+c > b+d$

ii) $a < b \Rightarrow a + c < b + c$ ii) $a < b \wedge c < d \Rightarrow a+c < b+d$

4) Multiplicative Properties:

a) $\forall\ a,b,c \in \Re$ and $c > 0$

i) $a > b \Rightarrow ac > bc$ ii) $a < b \Rightarrow ac < bc$.

b) $\forall\ a,b,c \in \Re$ and $c < 0$.

i) $a > b \Rightarrow ac < bc$ ii) $a < b \Rightarrow ac > bc$

c) $\forall\ a,b,c,d \in \Re$ and a,b,c,d are all positive.

i) $a > b \wedge c > d \Rightarrow ac > bd$. ii) $a < b \wedge c < d \Rightarrow ac < bd$

Note That:

1. Any set possessing all the above 11 properties is called a field.
2. From the multiplicative properties of inequality we conclude that: - If both the sides of an inequality are multiplied by a +ve number, its direction does not change, but multiplication of the two sides by -ve number reverses the direction of the inequality.
3. a and $(-a)$ are additive inverses of each other. Since by definition inverse of $-a$ is a,

$$\therefore\quad -(-a) = a$$

4. The left hand member of the above equation should be read as negative of 'negative a' and not 'minus minus a'.
5. a and $\frac{1}{a}$ are the multiplicative inverses of each other. Since by definition inverse of $\frac{1}{a}$ is a (i.e., inverse of a^{-1} is a), $a \neq 0$

$$\therefore\quad (a^{-1})^{-1} = a \quad \text{or} \quad \frac{1}{\frac{1}{a}} = a$$

Example 4: Prove that for any real numbers a, b

i) $a.0 = 0$ ii) $ab = 0 \Rightarrow a = 0 \vee b = 0$ [$\vee$ stands for "or"]

Solution: i) $a.0 = a[1 + (-1)]$ (Property of additive inverse)

$= a(1-1)$ (Def. of subtraction)

$= a.1 - a.1$ (Distributive Law)

$= a - a$ (Property of multiplicative identity)

$= a + (-a)$ (Def. of subtraction)

$= 0$ (Property of additive inverse)

Thus $a.0 = 0$.

ii) Given that $ab = 0$ (1)

Suppose $a \neq 0$, then exists

(1) gives: $\frac{1}{a}(ab) = \frac{1}{a}.0$ (Multiplicative property of equality)

$\Rightarrow (\frac{1}{a}.a)b = \frac{1}{a}.0$ (Assoc. law of $\times$)

$\Rightarrow 1.b = 0$ (Property of multiplicative inverse).

$\Rightarrow b = 0$ (Property of multiplicative identity).

Thus if $ab = 0$ and $a \neq 0$, then $b = 0$

Similarly it may be shown that

if $ab = 0$ and $b \neq 0$, then $a = 0$.

Hence $ab = 0 \Rightarrow a = 0$ or $b = 0$.

Example 5: For real numbers a,b show the following by stating the properties used.

i) $(-a)\, b = a\,(-b) = -ab$ ii) $(-a)\,(-b) = ab$

Solution: i) $(-a)(b) + ab = (-a + a)b$ (Distributive law)

$= 0.b = 0.$ (Property of additive inverse)

$\therefore\ (-a)b + ab = 0$

i.e.. $(-a)b$ and ab are additive inverse of each other.

$\therefore\ (-a)b = -(ab) = -ab$ (Θ $-(ab)$ is written as $-ab$)

ii) $(-a)\,(-b) - ab = (-a)(-b) + (-ab)$

$= (-a)(-b) + (-a)(b)$ (By (i))

$= (-a)(-b + b)$ (Distributive law)

$= (-a)\,.0 = 0.$ (Property of additive inverse)

$(-a)(-b) = ab$

Example 6: Prove that

i) $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$ (Principle for equality of fractions

ii) $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$

iii) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ (Rule for product of fractions).

iv) $\frac{a}{b} = \frac{ka}{kb}, (k \neq 0)$ (Golden rule of fractions)

v) $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$ (Rule for quotient of fractions).

The symbol $\Leftrightarrow$ stands for iff i.e.. if and only if.

Solution:

i) $\frac{a}{b} \Rightarrow \frac{c}{d}$ $\frac{a}{b}(bd)$ $\frac{c}{d}(bd)$

$\Rightarrow \frac{a.1}{b}(bd) = \frac{c.1}{d}(bd)$

$\Rightarrow a.(\frac{1}{b}.b).d = c.(\frac{1}{d}.bd)$

$= c(bd.\frac{1}{d})$

$\Rightarrow ad = cb$

$\therefore\ ad = bc$

Again $ad = bc \Rightarrow (ad) \times \frac{1}{b}.\frac{1}{d} = b.c.\frac{1}{b}.\frac{1}{d}$

$\Rightarrow a.\frac{1}{b}.d\frac{1}{d} = b.\frac{1}{b}\ .\ c.\frac{1}{d}$

$\Rightarrow \frac{a}{b} = \frac{c}{d}.$

ii) $(ab).\frac{1}{a}.\frac{1}{b} = (a.\frac{1}{a}).(b\frac{1}{b}) = 1.1 = 1$

Thus ab and $\frac{1}{a}.\frac{1}{b}$ are the multiplicative inverse of each other. But multiplicative inverse of ab is $\frac{1}{ab}$

$\therefore \frac{1}{ab} = \frac{1}{a}.\frac{1}{b}.$

iii) $\frac{a}{b}.\frac{c}{d} = (a.\frac{1}{b}).(c.\frac{1}{d})$

$$=(ac)(\frac{1}{b}.\frac{1}{d})$$ (Using commutative and associative laws of multiplication)

$$=ac.\frac{1}{bd}=\frac{ac}{bd}.$$
$$=\frac{a}{b}.\frac{c}{d}=\left|\frac{ac}{bd}\right|$$

iv) $\frac{a}{b}=\frac{a}{b}.1=\frac{a}{b}.\frac{k}{k}=\frac{ak}{ak}$

$\therefore \frac{a}{b}=\frac{ak}{bk}.$

v) $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{\frac{a}{b}(bd)}{\frac{c}{d}(bd)}=\frac{ad(\frac{1}{b}.b)}{cb(\frac{1}{d}.d)}=\frac{ad}{bc}.$

Example 7: Does the set {1, –1 } possess closure property with respect to
i) addition ii) multiplication?

Solution: i) 1 + 1 = 2, 1 + (–1) = 0 = –1 + 1
–1 + (–1) = –2

But 2, 0, –2 do not belong to the given set. That is, all the sums do not belong to the given set. So it does not possess closure property w.r.t. addition.

ii) 1.1= 1, 1. (–1) = –1, (–1) .1 = –1, (–1). (–1) = 1

Since all the products belong to the given set, it is closed w.r.t multiplication.

Exercise 1.1

1. Which of the following sets have closure property w.r.t. addition and multiplication?

i) {0} ii) {1} iii) (0 , – 1) iv) { 1, –1 }

2. Name the properties used in the following equations.
(Letters, where used, represent real numbers).

i) 4 + 9 = 9 + 4 ii) $(a+1)+\frac{3}{4}=a+(1+\frac{3}{4})$

iii) $(\sqrt{3}+\sqrt{5})+\sqrt{7}=\sqrt{3}+(\sqrt{5}+\sqrt{7})$ iv) 100 + 0 = 100

v) 1000 × 1 = 1000 vi) 4.1 + (– 4 .1) = 0

vii) $a - a = 0$ viii)

ix) $a(b - c) = ab - ac$ x) $(x - y)z = xz - yz$

xi) 4×(5 × 8) = (4 × 5) × 8 xii) $a(b + c - d) = ab + ac - ad$.

3. Name the properties used in the following inequalities:

i) $-3 < -2 \Rightarrow 0 < 1$ ii) $-5 < -4 \Rightarrow 20 > 16$

iii) $1 > -1 \Rightarrow -3 > -5$ iv) $a < 0 \Rightarrow -a > 0$

v) $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$ vi) $a > b \Rightarrow -a < -b$

4. Prove the following rules of addition: -

i) $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ ii) $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

5. Prove that $-\frac{7}{12}-\frac{5}{18}=\frac{-21-10}{36}$

6. Simplify by justifying each step: -

i) $\frac{4+16x}{4}$ ii) $\frac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}$

iii) $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$ iv) $\dfrac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}.\frac{1}{b}}$

1.4 Complex Numbers

The history of mathematics shows that man has been developing and enlarging his concept of **number** according to the saying that "Necessity is the mother of invention". In the remote past they stared with the set of counting numbers and invented, by stages, the negative numbers, rational numbers, irrational numbers. Since square of a positive as well as negative number is a positive number, the square
root of a negative number does not exist in the realm of real numbers. Therefore, square roots of negative numbers were given no attention for centuries together. However, recently, properties of numbers involving square roots of negative numbers have also been discussed in detail and such numbers have been found useful and have been applied in many branches o f pure and applied mathematics. The numbers of the
form $x + iy$, where $x, y \in \Re$, and i = ,are called **complex numbers,** here x is called **real part** and y is called **imaginary part** of the complex

number. For example, 3 + 4i, 2 – i etc. are complex numbers.

Note: Every real number is a complex number with 0 as its imaginary part.

Let us start with considering the equation.

$$x^2 + 1 = 0 \qquad (1)$$

$$\Rightarrow \quad x^2 = -1$$

$$\Rightarrow \quad x = \pm\sqrt{-1}$$

$\sqrt{-1}$ does not belong to the set of real numbers. We, therefore, for convenience call it **imaginary number** and denote it by i (read as iota).

The product of a real number and i is also an **imaginary number**

Thus $2i$, $-3i$, $\sqrt{5}i$, $-\frac{11}{2}i$ are all imaginary numbers, i which may be written $1.i$ is also an imaginary number.

Powers of i :

$i^2 = -1$ (by defination)

$i^3 = i^2.i = -1.i = -i$

$i^4 = i^2 \times i^2 = (-1)(-1) = 1$

Thus any power of i must be equal to 1, i,–1 or $-i$. For instance,

$i^{13} = (i^2)^6.i = (-1)^6.i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1$ etc.

1.4.1 Operations on Complex Numbers

With a view to develop algebra of **complex numbers**, we state a few definitions. The symbols a,b,c,d,k, where used, represent real numbers.

1) $a + bi = c + di \Rightarrow a = c \;\; b = d$.

2) Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$

3) $k(a + bi) = ka + kbi$

4) $(a + bi) - (c + di) = (a + bi) + [-(c + di)]$

$= a + bi + (-c - di)$

$= (a - c) + (b - d)i$

5) $(a + bi).(c + di) = ac + adi + bci + bdi = (ac - bd) + (ad + bc)i$.

6) Conjugate Complex Numbers: Complex numbers of the form (a + bi) and (a – bi) which have the same real parts and whose imaginary parts differ in sign only, are called conjugates of each other. Thus $5 + 4i$ and $5 - 4i$, $-2 + 3i$ and $-2 - 3i$,$-\sqrt{5}\,i$ and $\sqrt{5}\,i$ are three pairs of conjugate numbers.

Note: A real number is self-conjugate.

1.4.2 Complex Numbers as Ordered Pairs of Real Numbers

We can define complex numbers also by using ordered pairs. Let C be the set of ordered pairs belonging to $\Re \times \Re$ which are subject to the following properties: -

i) $(a,b)=(c,d) \Leftrightarrow a=c \wedge b=d.$
ii) $(a, b) + (c, d) = (a + c, b + d)$
iii) If k is any real number, then $k(a, b) = (ka, kb)$
iv) $(a, b)(c, d) = (ac - bd, ad + bc)$

Then C is called the set of *complex numbers*. It is easy to sec that $(a, b) - (c, d) = (a - c, b - d)$

Properties (1), (2) and (4) respectively define equality, sum and product of two complex numbers. Property (3) defines the product of a real number and a complex number.

Example 1: Find the sum, difference and product of the complex numbers (8, 9) and (5, –6)

Solution: Sum = (8 + 5, 9 – 6) = (13, 3)
Difference = (8 – 5, 9 – (–6)) = (3, 15)
Product = (8.5 – (9)(–6), 9.5 + (–6) 8)
= (40 + 54, 45 – 48)
= (94, –3)

1.4.3 Properties of the Fundamental Operations on Complex Numbers

It can be easily verified that the set C satisfies all the field axioms i.e., it possesses the properties 1(i to v), 2(vi to x) and 3(xi) of Art. 1.3.

By way of explanation of some points we observe as follows:-

i) The additive identity in C is (0, 0).
ii) Every complex number (a, b) has the additive inverse $(-a, -b)$ i.e., $(a, b) + (-a, -b) = (0, 0)$.
iii) The multiplicative identity is (1, 0) i.e.,
$(a, b).(1, 0) = (a.1 - b.0, b.1 + a.0) = (a, b)$.
$= (1, 0)(a, b)$
iv) Every non-zero complex number {i.e., number not equal to (0, 0)} has a multiplicative inverse.

The multiplicative inverse of (a, b) is $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

$$(a,b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1, 0), \text{ the identity element}$$
$$= \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)(a,b)$$

v) $(a, b)[(c, d) \pm (e, f)] = (a, b)(c, d) \pm (a, b)(e, f)$

Note: The set C of complex numbers does not satisfy the order axioms. In fact there is no sense in saying that one complex number is greater or less than another.

1.4.4 A Special Subset of *C*

We consider a subset of C whose elements are of the form $(a, 0)$ i.e., second component of each element is zero.

Let $(a, 0)$, $(c, 0)$ be two elements of this subset. Then

i) $(a, 0) + (c, 0) = (a + c, 0)$ ii) $k(a, 0) = (ka, 0)$
iii) $(a, 0) \times (c, 0) = (ac, 0)$
iv) Multiplicative inverse of $(a, 0)$ is $\left(\frac{1}{a}, 0\right), a \neq 0.$

Notice that the results are the same as we should have obtained if we had operated on the real numbers a and c ignoring the second component of each ordered pair i.e., 0 which has played no part in the above calculations.

On account of this special feature wc identify the complex number $(a, 0)$ with the real number a i.e., we postulate:

$(a, 0) = a$ (1)
Now consider (0, 1)
$(0, 1) . (0, 1) = (-1, 0)$
$= -1$ (by (1) above).

If we set $(0, 1) = i$ (2)
then $(0, 1)^2 = (0,1)(0,1) = i.i = i^2 = -1$

We are now in a position to write every complex number given as an ordered pair, in terms of i. For example

$(a, b) = (a, 0) + (0, b)$ (def. of addition)

$= a(1, 0) + b(0, 1)$ (by (1) and (2) above)
$= a.1 + bi$
$= a + ib$

Thus $(a, b) = a + ib$ where $i^2 = -1$

This result enables us to convert any Complex number given in one notation into the other.

Exercise 1.2

1. Verify the addition properties of complex numbers.
2. Verify the multiplication properties of the complex numbers.
3. Verify the distributive law of complex numbers.

$(a, b)[(c, d) + (e, f)] = (a, b)(c, d) + (a, b)(e, f)$

(**Hint:** Simplify each side separately)

4. Simplify' the following:

i) i^9 ii) i^{14} iii) $(-i)^{19}$ iv) $(-\frac{21}{2})$

5. Write in terms of i

i) $\sqrt{-1b}$ ii) $\sqrt{-5}$ iii) $\sqrt{\frac{-16}{25}}$ iv) $\sqrt{\frac{1}{-4}}$

Simplify the following:

6. $(7, 9) + (3, -5)$ **7.** $(8, -5) - (-7, 4)$ **8.** $(2, 6)(3, 7)$
9. $(5, -4)(-3, -2)$ **10.** $(0, 3)(0, 5)$ **11.** $(2, 6) \div (3, 7)$.

12. $(5, -4) \div (-3, -8)$ $\left(\text{Hint for 11: } \frac{(2,6)}{(3,7)} = \frac{2+6i}{3+7i} \times \frac{3-7i}{3-7i} \text{ etc.}\right)$

13. Prove that the sum as well as the product of any two conjugate complex numbers is a real number.

14. Find the multiplicative inverse of each of the following numbers:

i) $(-4, 7)$ ii) $(\sqrt{2}, -\sqrt{5})$ iii) $(1, 0)$

15. Factorize the following:

i) $a^2 + 4b^2$ ii) $9a^2 + 16b^2$ iii) $3x^2 + 3y^2$

16. Separate into real and imaginary parts (write as a simple complex number): -

i) $\frac{2-7i}{4+5i}$ ii) $\frac{(-2+3i)^2}{(1+i)}$ iii) $\frac{i}{1+i}$

1.5 The Real Line

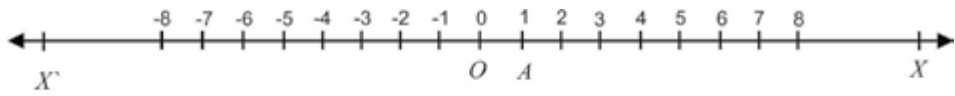



Fig. (1)

In Fig.(1), let $\overleftrightarrow{X'X}$ be a line. We represent the number 0 by a point O (called the origin) of the line. Let $|OA|$ represents a unit length. According to this unit, positive numbers are represented on this line by points to the right of O and negative numbers by points to the left of O. It is easy to visualize that all +ve and –ve rational numbers are represented on this line. What about the irrational numbers?

The fact is that all the irrational numbers are also represented by points of the line. Therefore, we postulate: -

Postulate: A (1 – 1) correspondence can be established between the points of a line ℓ and the real numbers in such a way that:-

i) The number 0 corresponds to a point O of the line.
ii) The number 1 corresponds to a point A of the line.
iii) If x_1, x_2 are the numbers corresponding to two points P_1, P_2, then the distance between P_1 and P_2 will be $|x_1 - x_2|$.

It is evident that the above correspondence will be such that corresponding to any real number there will be one and only one point on the line and vice versa.

When a (1 – 1) correspondence between the points of a line $x'x$ and the real numbers has been established in the manner described above, the line is called the **real line** and the real number, say x, corresponding to any point P of the line is called the **coordinate** of the point.

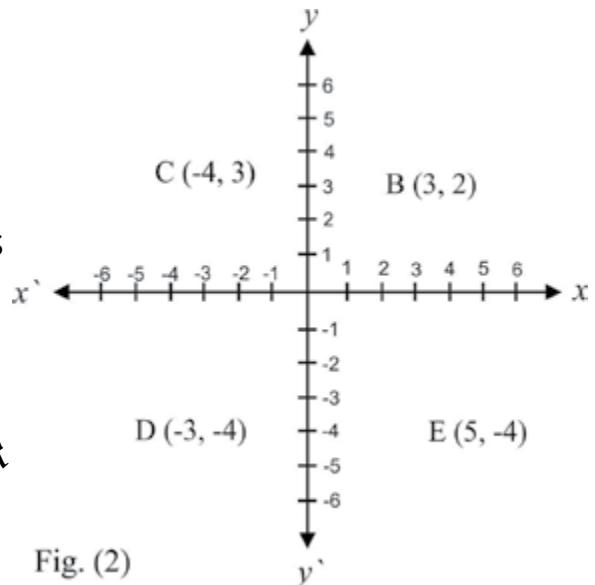
1.5.1 The Real Plane or The Coordinate Plane

We know that the *cartesian product* of two non-empty sets A and B, denoted by $A \times B$, is the set: $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

The members of a *cartesian* product are **ordered pairs.**

The cartesian product $\Re\times\Re$ where $\Re$ is the set of real numbers is called the **cartesian plane**.

By taking two perpendicular lines $x'ox$ and $y'oy$ as coordinate axes on a geometrical plane and choosing a convenient unit of distance, elements of $\Re\times\Re$ can be represented on the plane in such a way that there is *a* (1–1) correspondence between the elements of $\Re\times\Re$ and points of the plane.

Fig. (2)

The geometrical plane on which coordinate system has been specified is called the **real plane** or the **coordinate plane**.

Ordinarily we do not distinguish between the Cartesian plane $\Re\times\Re$ and the coordinate plane whose points correspond to or represent the elements of $\Re\times\Re$.

If a point *A* of the coordinate plane corresponds to the ordered pair (*a*, *b*) then *a*, *b* are called the **coordinates** of *A*. *a* is called the *x* - coordinate or **abscissa** and *b* is called the y - coordinate or **ordinate**.

In the figure shown above, the coordinates of the points*B, C, D* and *E* are (3, 2), (–4, 3), (–3, –4) and (5, –4) respectively.

Corresponding to every ordered pair (*a, b*) $\in\Re\times\Re$ there is one and only one point in the plane and corresponding to every point in the plane there is one and only one ordered pair (*a, b*) in $\Re\times\Re$.

There is thus a (1 – 1) correspondence between $\Re\times\Re$ and the plane.

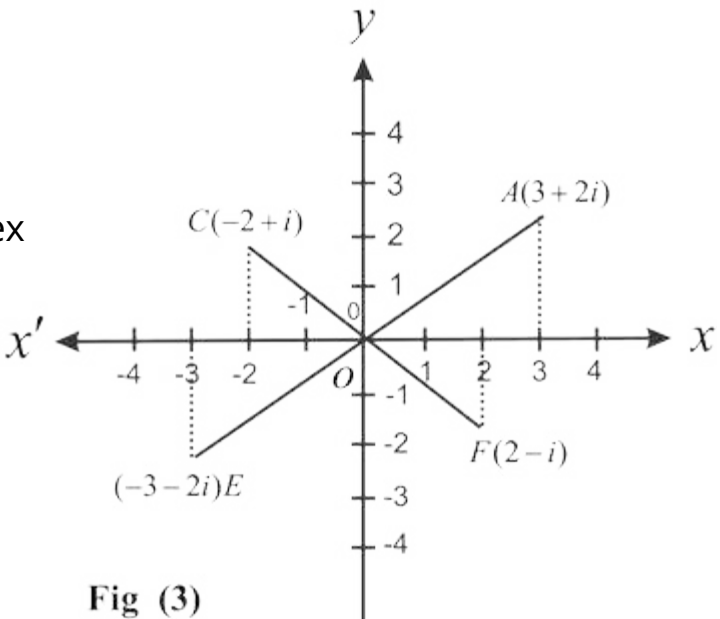
1.6 Geometrical Representation of Complex Numbers The Complex Plane

We have seen that there is a (1–1) correspondence between the elements (ordered pairs) of the Cartesian plane $\Re\times\Re$ and the complex numbers. Therefore, there is a (1– 1) correspondence between the points of the coordinate plane and the complex numbers. We can, therefore, represent complex numbers by points of the coordinate plane. In this representation every complex number will be represented by one and only one point of

the coordinate plane and every point of the plane will represent one and only one complex number. The components of the complex number will be the coordinates of the point representing it. In this representation the ***x*-axis** is called the real axis and the ***y*-axis** is called the **imaginary axis**. The coordinate plane itself is called the **complex plane** or **z – plane**.

By way of illustration a number of complex numbers have been shown in figure 3.

The figure representing one or more complex numbers on the *complex plane* is called an **Argand diagram**. Points on the **x-axis** represent **real numbers** whereas the points on the **y-axis** represent **imaginary numbers.**

Fig (3)

In fig (4), *x*, *y* are the coordinates of a point.

It represents the complex number *x* + *iy*.

The *real number* $\sqrt{x^2+y^2}$ is called **the modulus of the complex number *a* + *ib*.**

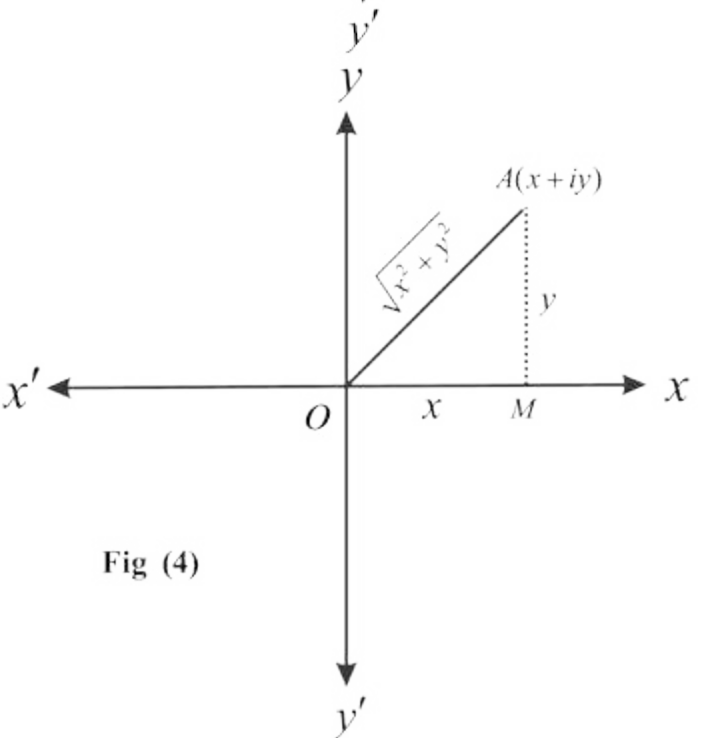
In the figure $\overline{MA}\perp\overrightarrow{ox}$

$\therefore \overline{OM}=x,\ \overline{MA}=y$

In the right-angled triangle *OMA*, we have, by Pythagoras theorem,

$$|\overline{OA}|^2=|\overline{OM}|^2+|\overline{MA}|^2$$

$$\therefore |\overline{OA}|=\sqrt{x^2+y^2}$$

Fig (4)

Thus $|\overline{OA}|$ represents the modulus of *x* + *iy*. In other words: **The modulus of a complex number is the distance from the origin of the point representing the number.**

The modulus of a complex number is generally denoted as: $|x+iy|$ or $|(x, y)|$. For convenience, a complex number is denoted by z.

If $z = x + iy = (x, y)$, then

$$|z| = \sqrt{x^2 + y^2}$$

Example 1: Find moduli of the following complex numbers :

(i) $1-i\sqrt{3}$ (ii) 3 (iii) $-5i$ (iv) $3 + 4i$

Solution:

i) Let $z = 1 - i\sqrt{3}$

or $z = 1 + i\left(-\sqrt{3}\right)$

$$\therefore |z| = \sqrt{(1)^2 + (-\sqrt{3})^2}$$

$$= \sqrt{1+3} = 2$$

ii) Let $z = 3$

or $z = 3 + 0.i$

$$\therefore |z| = \sqrt{(3)^2 + (0)^2} = 3$$

iii) Let $z = -5i$

or $z = 0 + (-5)i$

$$\therefore |z| = \sqrt{0^2 + (-5)^2} = 5$$

iv) Let $z = 3 + 4i$

$$\therefore |z| = \sqrt{(3)^2 + (4)^2}$$

Theorems: $\forall z, z_1, z_2 \in C$,

i) $|-z| = |z| = |\bar{z}| = |-\bar{z}|$

ii) $\bar{\bar{z}} = z$

iii) $z\bar{z} = |z|^2$

iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

v) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

vi) $|z_1.z_2| = |z_1|.|z_2|$

Proof :(i): Let $z = a + ib$,

So, $-z = -a - ib, \bar{z} = a - ib$ and $-\bar{z} = -a + ib$

$$\therefore |-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} \quad (1)$$

$$|z| = \sqrt{a^2 + b^2} \quad (2)$$

$$|\bar{z}| = \sqrt{(a)^2 + (b)^2} = \sqrt{a^2 + b^2} \quad (3)$$

$$|-\bar{z}| = \sqrt{(-a)^2 + (b)^2} = \sqrt{a^2 + b^2} \quad (4)$$

By equations (1), (2), (3) and (4) we conclude that

$$|-z| = |z| = |\bar{z}| = |-\bar{z}|$$

(ii) Let $z = a + ib$

So that $\bar{z} = a - ib$

Taking conjugate again of both sides, we have

$$\bar{\bar{z}} = a + ib = z$$

(iii) Let $z = a + ib$ so that $\bar{z} = a - ib$

$$\therefore z.\bar{z} = (a+ib)(a-ib)$$
$$= a^2 - iab + iab - i^2b^2$$
$$= a^2 - (-1)b^2$$
$$= a^2 + b^2 = |z|^2$$

(iv) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$z_1 + z_2 = (a + ib) + (c + id)$

$= (a + c) + i(b + d)$

so, $\overline{z_1 + z_2} = \overline{(a+c) + i(b+d)}$ (Taking conjugate on both sides)

$= (a + c) - i(b + d)$

$= (a - ib) + (c - id) = \bar{z}_1 + \bar{z}_2$

(v) Let $z_1 = a + ib$ and $z_2 = c + id$, where $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$$

$$= \frac{a+ib}{c+id} \times \frac{c-id}{c-id} \qquad \text{(Note this step)}$$

$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

$$\therefore \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}}$$

$$= \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2} \qquad (1)$$

Now $\quad \frac{\bar{z}_1}{\bar{z}_2} = \frac{\overline{a+ib}}{\overline{c+id}} = \frac{a-ib}{c-id}$

$$= \frac{a-ib}{c-id} \times \frac{c+id}{c+id}$$

$$= \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2} \qquad (2)$$

From (1) and (2), we have

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

(vi) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$|z_1.z_2| = |(a+ib)(c+id)|$$

$$= |(ac-bd)+(ad+bc)i|$$

$$= \sqrt{(ac-bd)^2+(ad+bc)^2}$$

$$= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$$

$$= \sqrt{(a^2+b^2)(c^2+d^2)}$$

$$= |z_1|.|z_2|$$

This result may be stated thus: -

The modulus of the product of two complex numbers is equal to the product of their moduli.

(vii) Algebraic proof of this part is tedius. Therefore, we prove it geometrically.

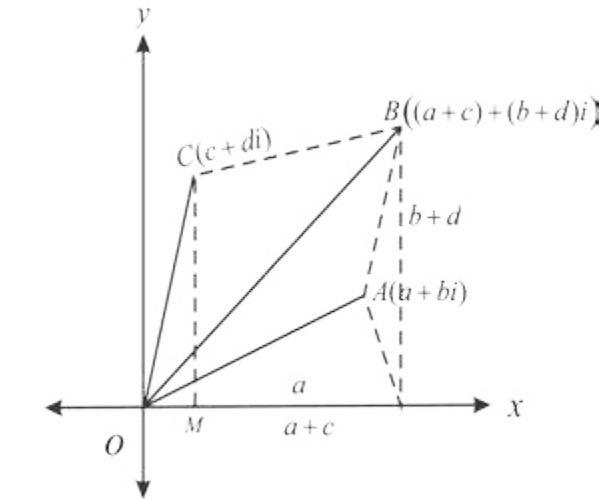



Fig (5)

In the figure point *A* represents $z_1 = a + ib$ and point *C* represents $z_2 = c + id$. We complete the parallelogram *OABC*. From the figure, it is evident that coordinates of *B* are (*a* + *c*, *b* + *d*), therefore, *B* represents

$z_1 + z_2 = (a + c) + (b + d)i$ and $|\overline{OB}| = |z_1 + z_2|$.

Also $|\overline{OA}| = |z_1|$, $\qquad |\overline{AB}| = |\overline{OC}| = |z_2|$.

In the $\triangle OAB$; $OA + AB > OB$ $\qquad$ ($OA = m\overline{OA}$ etc.)

$$\therefore |z_1| + |z_2| > |z_1 + z_2| \qquad (1)$$

Also in the same triangle, $OA - AB < OB$

$$\therefore |z_1| - |z_2| < |z_1 + z_2| \qquad (2)$$

Combining (1) and (2), we have

$$|z_1| - |z_2| < |z_1 + z_2| < |z_1| + |z_2| \qquad (3)$$

which gives the required results with inequality signs.

Results with equality signs will hold when the points *A* and *C* representing z_1 and z_2 become collinear with *B*. This will be so when $\frac{a}{b}=\frac{c}{d}$ (see fig (6)).

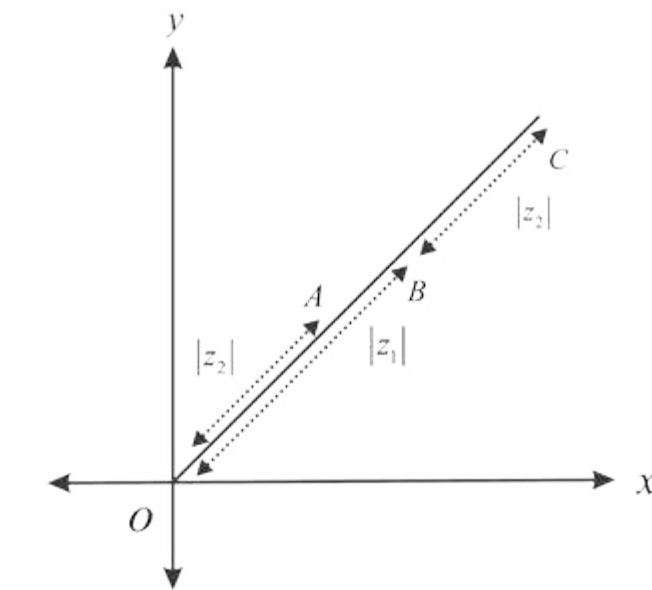



Fig (6)

In such a case $|z_1|+|z_2|=|\overline{OB}|+|\overline{OA}|$

$$=|\overline{OB}|+|\overline{BC}|$$
$$=|\overline{OC}|$$
$$=|z_1+z_2|$$

Thus $\quad |z_1+z_2|=|z_1|+|z_2|$

The second part of result (vii) namely

$$|z_1+z_2|\le|z_1|+|z_2|$$

is analogue of the triangular inequality*. In words, it may be stated thus: - The modulus of the sum of two complex numbers is less than or equal to the sum of the moduli of the numbers.

Example 2: If z_1 = 2+ *i*, z_2 = 3 – 2*i*, z_3 = 1 + 3*i* then express $\frac{\overline{z_1}\ \overline{z_3}}{z_2}$ in the form *a* + *ib*

(Conjugate of a complex number *z* is denoted as $\overline{z}$)

Solution:

$$\frac{\overline{z_1}\ \overline{z_3}}{z_2}=\frac{\overline{(2+i)}\ \overline{(1+3i)}}{3-2i}=\frac{(2-i)\ (1-3i)}{3-2i}$$

$$=\frac{(2-3)+(-6-1)i}{3-2i}=\frac{-1-7i}{3-2i}$$
$$=\frac{(-1-7i)(3+2i)}{(3-2i)\ (3+2i)}$$
$$=\frac{(-3+14)+(-2-21)i}{3^2+2^2}=\frac{11}{13}\ \ \frac{23}{13}i$$

Example 3: Show that, $\forall\ z_1, z_2 \in C,\ \overline{z_1\ z_2}=\overline{z_1}\ \overline{z_2}$

Solution: Let $z_1 = a + bi,\ z_2 = c + di$

$$\overline{z_1z_2}=\overline{(a+bi)(c+di)}=\overline{(ac-bd)(ad+bc)i}$$
$$=(ac-bd)-(ad+bc)i \qquad (1)$$

$$\overline{z_1}.\overline{z_2}=\overline{(a+bi)}=\overline{(c+di)}$$
$$=(a-bi)(c-di)$$
$$=(ac-bd)+(-ad-bc)i$$

$$\overline{z_1}.\overline{z_2}=\overline{(a+bi)}\ \overline{(c+di)}$$
$$=(a-bi)(c-di)$$
$$=(ac-bd)+(-ad-bc)i \qquad (2)$$

Thus from (1) and (2) we have, $\overline{z_1\ z_2}=\overline{z_1}\ \overline{z_2}$

Polar form of a Complex number: Consider adjoining diagram representing the complex number *z* = *x* + *iy*. From the diagram, we see that *x* = *r*cosθ and *y* = *r*sinθ where *r* = |*z*| and θ is called argument of *z*.

Hence $\quad$ *x* + *iy* = *r*cosθ + *r*sinθ $\quad$ ....(i)

where $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\frac{y}{x}$

Equation (i) is called the polar form of the complex number *z*.

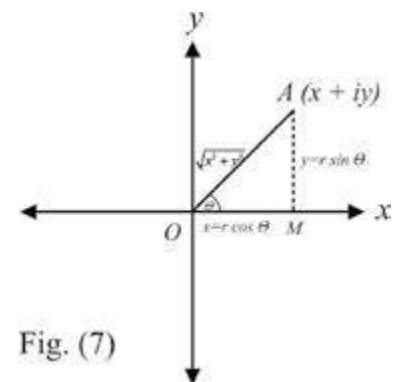



Fig. (7)

*In any triangle the sum of the lengths of any two sides is greater than the length of the third side and difference of the lengths of any two sides is less than the length of the third side.

Example 4: Express the complex number $1+i\sqrt{3}$ in polar form.

Solution:

Step-I: Put $r\cos\theta = 1$ and $r\sin\theta = \sqrt{\ }$

Step-II: $r^2 = (1)^2 + (\sqrt{3})^2$

$\Rightarrow r^2 = 1+3 = 4 \quad \Rightarrow r = 2$

Step-III: $\theta = \tan^{-1}\frac{\sqrt{3}}{1} = \tan^{-1}\sqrt{3} = 60^\circ$

Thus $1+i\sqrt{3} = 2\cos 60^\circ + i2\sin 60^\circ$

De Moivre's Theorem : -

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \ \forall n \in Z$

Proof of this theorem is beyond the scope of this book.

1.7 To find real and imaginary parts of

i) $(x + iy)^n$ **ii)** $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n, \ x_2+iy_2 \neq 0$

for $n = \pm1, \pm2, \pm3, \ldots$

i) Let $x = r\cos\theta$ and $y = r\sin\theta$, then

$(x + iy)^n = (r\cos\theta + ir\sin\theta)^n$

$= (r\cos\theta + ir\sin\theta)^n$

$= [r(\cos\theta + i\sin\theta)]^n$

$= r^n(\cos\theta + i\sin\theta)^n$

$= r^n(\cos n\theta + i\sin n\theta)$ (By De Moivre's Theorem)

$= r^n \cos n\theta + ir^n \sin n\theta$

Thus $r^n \cos n\theta$ and $r^n \sin n\theta$ are respectively the real and imaginary parts of $(x + iy)^n$.

Where $r = \sqrt{x^2 \quad y^2}$ and $\theta \quad tan^{-1}\frac{x}{y}$.

ii) Let $x_1 + iy_1 = r_1\cos\theta_1 + r_1\sin n\theta_1$ and $x_2 + iy_2 = r_2\cos\theta_2 + r_2\sin n\theta_2$ then,

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{r_1\cos\theta_1 + r_1\, i\sin\theta_1}{r_2\cos\theta_2 + r_2\, i\sin\theta_2}\right)^n \quad \frac{r_1^{\,n}(\cos\theta_1 + i\sin\theta_1)^n}{r_2^{\,n}(\cos\theta_2 + i\sin\theta_2)^n}$$

$$= \frac{r_1^{\,n}}{r_2^{\,n}}(\cos\theta_1 \quad i\sin\theta_1)^n(\cos\theta_2 \quad i\sin\theta_2)^{-n}$$

$$= \frac{r_1^{\,n}}{r_2^{\,n}}(\cos n\theta_1 + i\sin n\theta_1)\big(\cos(-n\theta)_2 + i\sin(-n\theta_2)\big),$$

(By De Moivre's Theorem)

$$= \frac{r_1^{\,n}}{r_2^{\,n}}(\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 - i\sin n\theta_2), \ (\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta)$$

$$= \frac{r_1^{\,n}}{r_2^{\,n}}.[(\cos n\theta_1 \cos n\theta_2 \quad \sin n\theta_1 \sin n\theta_2)$$

$$+ i(\sin n\theta_1 \cos n\theta_2 \ - \ \cos n\theta_1 \ \sin n\theta_2)]$$

$$= \frac{r_1^{\,n}}{r_2^{\,n}}[\cos(n\theta_1 - n\theta_2) + i\sin(n\theta_1 - n\theta_2)] \because \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\text{and } \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \frac{r_1^{\,n}}{r_2^{\,n}}[\cos n(\theta_1 \ - \ \theta_2) + i\sin n(\theta_1 - \theta_2)]$$

$$= \frac{r_1^{\,n}}{r_2^{\,n}}[\cos n(\theta_1 \quad \theta_2) + i\sin n\ (\theta_1 - \theta_2)]$$

Thus $\frac{r_1^{\,n}}{r_2^{\,n}}\cos n\,(\theta_1 - \theta_2)$ and $\frac{r_1^{\,n}}{r_2^{\,n}}\sin n(\theta_1 - \theta_2)$ are respectively the real and imaginary parts of $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n, \ x_2+iy_2 \neq 0$

where $r_1 = \sqrt{x_1^2 + y_1^2}\ ; \theta_1 = tan^{-1}\frac{y_1}{x_1}$ and $r_2 = \sqrt{x_2^2 \quad y_2^2}; \ \theta_2 = \tan^{-1}\frac{y_2}{x_2}$

Example 5: Find out real and imaginary parts of each of the following complex numbers.

i) $\left(\sqrt{3}+\mathrm{i}\right)^3$ ii) $\left(\dfrac{1-\sqrt{3}\,\mathrm{i}}{1+\sqrt{3}\,\mathrm{i}}\right)^5$

Solution:

i) Let $r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$ where

$$r^2 = (\sqrt{3})^2 + 1^2 \text{ or } r = \sqrt{3+1} = 2 \text{ and } \theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^o$$

So, $\left(\sqrt{3}+i\right)^3 = (r\cos\theta + ir\sin\theta)^3$

$= r^3(\cos 3\theta + i\sin 3\theta)$ (By De Moivre's Theorem)

$= 2^3\,(\cos 90^o + i\sin 90^o)$

$= 8\,(0 + i.1)$

$= 8i$

Thus 0 and 8 are respectively real and imaginary Parts of $\left(\sqrt{3}+\mathrm{i}\right)^3$.

ii) Let $r_1\cos\theta_1 = 1$ and $r_1\sin\theta_1 = -\sqrt{3}$

$$\Rightarrow r_1 = \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2 \text{ and } \theta_1 = \tan^{-1} -\frac{\sqrt{3}}{1} - 60^o$$

Also Let $r_2\cos\theta_2 = 1$ and $r_2\sin\theta_2 = \sqrt{3}$

$$\Rightarrow r_2 = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2 \text{ and } \theta_2 \tan^{-1}\frac{\sqrt{3}}{1} = 60^o$$

So, $\left(\dfrac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5 = \left[\dfrac{2\left(\cos(-60^o) + i\sin(-60^O)\right)}{2\left(\cos(60^o) + i\sin(60^O)\right)}\right]^5$

$$= \frac{\left(\cos(-60^\circ) + i\sin(-60^\circ)\right)^5}{\left(\cos(60^\circ) + i\sin(60^\circ)\right)^5}$$

$$= \left(\cos(-60^o) + i\sin(-60^O)\right)^5 \left(\cos(60^o) + i\sin(60^O)\right)^{-5}$$

$$= \left(\cos(-300^o) + i\sin(-300^o)\right)\left(\cos(-300^o) + i\sin(-300^o)\right)$$

$$= \left(\cos(300^o)\quad i\sin(300^o)\right)\left(\cos(300^o)\quad i\sin(300^o)\right) - \because \cos(\quad\theta) = \cos\theta$$
$$\text{and } \sin(-\theta) = -\sin\theta$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$$

Thus $\dfrac{-1}{2}$, $\dfrac{\sqrt{3}}{2}$ are respectively real and imaginary parts of $\left(\dfrac{1-\sqrt{3}\,\mathrm{i}}{1+\sqrt{3}\,\mathrm{i}}\right)^5$

Exercise 1.3

1. Graph the following numbers on the complex plane: -

i) $2+3i$ ii) $2-3i$ iii) $-2-3i$ iv) $-2+3i$

v) -6 vi) i vii) $\frac{3}{5}-\frac{4}{5}i$ viii) $-5-6i$

2. Find the multiplicative inverse of each of the following numbers: -

i) $-3i$ ii) $1-2i$ iii) $-3-5i$ iv) (1, 2)

3. Simplify

i) i^{101} ii) $(-ai)^4$, $a \in \Re$ iii) i^{-3} iv) i^{-10}

4. Prove that $\bar{z} = z$ iff z is real.

5. Simplify by expressing in the form $a + bi$

i) $5 + 2\sqrt{-4}$ ii) $(2+\sqrt{-3})(3+\sqrt{-3})$

iii) $\dfrac{2}{\sqrt{5}+\sqrt{-8}}$ iv) $\dfrac{3}{\sqrt{6}-\sqrt{-12}}$

6. Show that $\forall z \in C$

i) $z^2 - \bar{z}^2$ is a real number. ii) $(z-\bar{z})^2$ is a real number.

7. Simplify the following

i) $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$

ii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3$

iiii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

iv) $(a+bi)^2$

v) $(a+bi)^{-2}$

vi) $(a+bi)^3$

vii) $(a-bi)^3$

viii) $(3-\sqrt{-4})^{-3}$

CHAPTER

2

Sets Functions and Groups

Animation 2.1: Function
Source & Credit: elearn.punjab

2.1 Introduction

We are familiar with the notion of a **set** since the word is frequently used in everyday speech, for instance, water set, tea set, sofa set. It is a wonder that mathematicians have developed this ordinary word into a mathematical concept as much as it has become a language which is employed in most branches of modern mathematics.

For the purposes of mathematics, a set is generally described as a **well-defined collection of distinct objects.** By a *well-defined collection* is meant a collection, which is such that, given any object, we may be able to decide whether the object belongs to the collection or not. By *distinct objects* we mean objects no two of which are identical (same).

The objects in a set are called its **members** or **elements**. Capital letters *A, B, C, X, Y, Z* etc., are generally used as names of sets and small letters *a, b, c, x, y, z* etc., are used as *members* of sets.

There are three different ways of describing a set

i) The Descriptive Method: A set may be described in words. For instance, the set of all vowels of the English alphabets.

ii) The Tabular Method: A set may be described by listing its elements within brackets. If *A* is the set mentioned above, then we may write:

$$A = \{a,e,i,o,u\}.$$

iii) Set-builder method: It is sometimes more convenient or useful to employ the method of set-builder notation in specifying sets. This is done by using a symbol or letter for an arbitrary member of the set and stating the property common to all the members.

Thus the above set may be written as:

$A = \{ x \mid x$ is a vowel of the English alphabet$\}$

This is read as *A* is the set of all x such that x is a vowel of the English alphabet.

The symbol used for **membership** of a set is $\in$. Thus $a \in A$ means ***a* is an element of *A*** or ***a* belongs to *A***. $c \notin A$ means ***c* does not belong to *A*** or ***c* is not a member of *A***. Elements of a set can be anything: people, countries, rivers, objects of our thought. In algebra we usually deal with sets of numbers. Such sets, alongwith their names are given below:-

N = The set of all natural numbers = $\{1,2,3,...\}$

W = The set of all whole numbers = $\{0,1,2,...\}$

Z = The set of all integers = $\{0,\pm1,+2....\}$.

Z' = The set of all negative integers = $\{-1,-2,-3,...\}$.

O = The set of all odd integers = $\{\pm1,\pm3,\pm5,...\}$.

E = The set of all even integers = $\{0,\pm2,\pm4,...\}$.

Q = The set of all rational numbers = $\left\{x \,\middle|\, x = \frac{p}{q} \text{ where p,q} \in Z \text{ and q} \neq 0\right\}$

Q' = The set of all irrational numbers = $\left\{x \,\middle|\, x \neq \frac{p}{q} \text{ where p,q} \in Z \text{ and q} \neq 0\right\}$

$\mathbb{R}$ = The set of all real numbers = $Q \cup Q'$

Equal Sets: Two sets *A* and *B* are equal i.e., *A*=*B*, if and only if they have the same elements that is, if and only if every element of each set is an element of the other set.

Thus the sets { 1, 2, 3 } and { 2, 1, 3} are equal. From the definition of equality of sets it follows that a mere change in the order of the elements of a set does not alter the set. In other words, while describing a set in the tabular form its elements may be written in any order.

Note: (1) *A* = *B* if and only if they have the same elements means if *A* = *B* they have the same elements and if *A* and *B* have the same elements then *A* = *B*.

(2) The phrase if and only if is shortly written as "iff ".

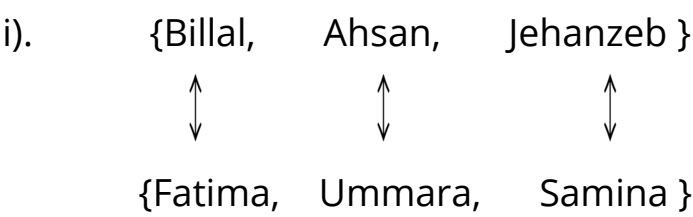
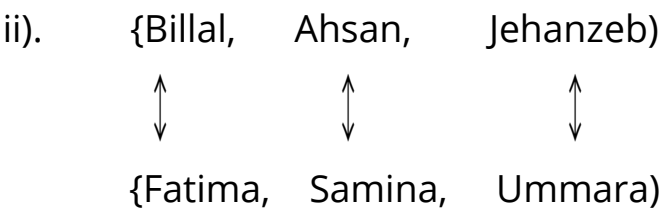
Equivalent Sets: If the elements of two sets *A* and *B* can be paired in such a way that each element of *A* is paired with one and only one element of *B* and vice versa, then such a pairing is called a one-to-one correspondence between *A* and *B* e.g., if *A* = {Billal, Ahsan, Jehanzeb} and *B* = {Fatima, Ummara, Samina} then six different (1 - 1) correspondences can be established between *A* and *B*

Two of these correspondences are shown below; -

i). {Billal, Ahsan, Jehanzeb }
↕ ↕ ↕
{Fatima, Ummara, Samina }

ii). {Billal, Ahsan, Jehanzeb)

$\updownarrow$ $\updownarrow$ $\updownarrow$

{Fatima, Samina, Ummara)

(Write down the remaining 4 correspondences yourselves)

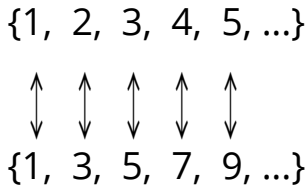
Two sets are said to be equivalent if *a* (1 – 1) correspondence can be established between them In the above example *A* and *B* are equivalent sets.

Example 1: Consider the sets *N*= {1, 2, 3,... } and *O* = {1, 3, 5,...}

We may establish (1–1) correspondence between them in the following manner:

{1, 2, 3, 4, 5, ...}

$\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$

{1, 3, 5, 7, 9, ...}

Thus the sets *N* and *O* are equivalent. But notice that they are not equal.

Remember that two equal sets are necessarily equivalent, but the converse may not be true i.e., two equivalent sets are not necessarily equal.

Sometimes, the symbol ~ is used to mean **is equivalent to**. Thus $N \sim O$.

Order of a Set: There is no restriction on the number of members of a set. A set may have 0, 1, 2, 3 or any number of elements. Sets with zero or one element deserve special attention. According to the everyday use of the word set or collection it must have at least two elements. But in mathematics it is found convenient and useful to consider sets which have only one element or no element at all.

A set having only one element is called a **singleton set** and a set with no element (zero number of elements) is called the **empty set** or **null set**.The empty set is denoted by the symbol ϕ or { }.The set of odd integers between 2 and 4 is a singleton i.e., the set {3} and the set of even integers between the same numbers is the empty set.

The solution set of the equation $x^2 + 1 = 0$, in the set of real numbers is also the empty set. Clearly the set {0} is a *singleton set* having zero as its only element, and not the empty set.

Finite and Infinite sets: If a set is equivalent to the set {1, 2, 3,...*n*} for some fixed natural number *n*, then the set is said to be *finite* otherwise *infinite*.

Sets of number *N, Z, Z'*etc., mentioned earlier are infinite sets.

The set {1, 3, 5,.......9999} is a finite set but the set { 1, 3, 5, ...}, which is the set of all positive odd natural numbers is an infinite set.

Subset: If every element of a set *A* is an element of set *B*, then *A* is a *subset* of *B*. Symbolically this is written as: $A \subseteq B$ (*A* is subset of *B*)

In such a case we say *B* is a super set of *A*. Symbolically this is written as:

$B \supseteq A$ {*B* is a superset of *A*)

Note: The above definition may also be stated as follows:

$A \subseteq B$ iff $x \in A \Rightarrow x \in B$

Proper Subset: If *A* is a subset of *B* and *B* contains at least one element which is not an element of *A*, then A is said to be a *proper subset* of *B*. In such a case we write: $A \subset B$ (*A* is a proper subset of *B*).

Improper Subset: If *A* is subset of *B* and *A* = *B*, then we say that *A* is an improper subset of *B*. From this definition it also follows that every set *A* is an improper subset of itself.

Example 2: Let *A* = { *a*, *b*, *c* }, *B* = { *c*, *a*, *b*} and *C* = { *a*, *b*, *c*, *d*}, then clearly

$A \subset C$, $B \subset C$ but $A = B$ and $B = A$.

Notice that each of *A* and *B* is an improper subset of the other because *A* = *B*

Note: When we do not want to distinguish between proper and improper subsets, we may use the symbol $\subseteq$ for the relationship. It is easy to see that: $N \subset Z \subset Q \subset \mathbb{R}$.

Theorem 1.1: The empty set is a subset of every set.

We can convince ourselves about the fact by rewording the definition of subset as follows: -

A is subset of *B* if it contains no element which is not an element of *B*.

Obviously an empty set does not contain such element, which is not contained by another set.

Power Set: A set may contain elements, which are sets themselves. For example if: *C* = Set of classes of a certain school, then elements of *C* are sets themselves because each class is a set of students. An important set of sets is the *power set* of a given set.

The *power set* of a set *S* denoted by *P* (*S*) is the set containing all the possible subsets of *S*.

Example 3: If $A = \{a, b\}$, then $P(A)=\{\Phi, \{a\}, \{b\}, \{a,b\}\}$
Recall that the empty set is a subset of every set and every set is its own subset.

Example 4: If $B = \{1, 2, 3\}$, then
$$P(B) = \{\Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Example 5: If $C = \{a, b, c, d\}$, then

Example 6: If $D = \{a\}$, then $P(D)=\{\Phi,\{a\}\}$

Example 7: If $E = \{\ \}$, then $P(E)=\{\Phi\}$

Note: (1) The power set of the empty set is not empty.
(2) Let *n* (*S*) denoted the number of elements of a set *S*, then *n*{*P*(*S*)} denotes the number of elements of the power set of *S*. From examples 3 to 7 we get the following table of results:

n(s)	0	1	2	3	4	5
n{p(s)}	$1=2^0$	$2=2^1$	$4=2^2$	$8=2^3$	$16=2^4$	$32=2^5$

In general if $n(S) = m$, then, $n\,P(S) = 2^m$

Universal Set: When we are studying any branch of mathematics the sets with which we have to deal, are generally subsets of a bigger set. Such a set is called the **Universal set** or the **Universe of Discourse**. At the elementary level when we are studying arithmetic, we have to deal with *whole numbers* only. At that stage the set of whole numbers can be treated as *Universal Set*. At a later stage, when we have to deal with negative numbers also and fractions, the set of the *rational numbers* can be treated as the *Universal Set*.

For illustrating certain concepts of the Set Theory, we sometimes consider quite

small sets (sets having small number of elements) to be universal. This is only an academic artificiality.

Exercise 2.1

1. Write the following sets in set builder notation:
   - i) { 1, 2, 3, ...,1000}
   - ii) { 0, 1,2,........., 100}
   - iii) { 0, ± 1, ± 2,............ ±1000}
   - iv) { 0, –1, –2,.........,–500}
   - v) {100, 101,102, ...,400}
   - vi) {–100,–101,–102,.., –500}
   - vii) {Peshawar, Lahore, Karachi, Quetta}
   - viii) { January, June, July }
   - xi) The set of all odd natural numbers
   - x) The set of all rational numbers
   - xi) The set of all real numbers between 1and 2,
   - xii) The set of all integers between – 100 and 1000

2. Write each of the following sets in the descriptive and tabular forms:-
   - i) $\{x | x \in N \wedge x \leq 10\}$
   - ii) $\{x | x \in N \wedge 4 < x < 12\}$
   - iii) $\{x | x \in Z \wedge -5 < x < 5\}$
   - iv) $\{x | x \in E \wedge 2 < x \leq 4\}$
   - v) $\{x | x \in P \wedge x < 12\}$
   - vi) $\{x | x \in O \wedge 3 < x < 12\}$
   - vii) $\{x | x \in E \wedge 4 \leq x \leq 10\}$
   - viii) $\{x | x \in E \wedge 4 < x < 6\}$
   - ix) $\{x | x \in O \wedge 5 \leq x \leq 7\}$
   - x) $\{x | x \in O \wedge 5 \leq x < 7\}$
   - xi) $\{x | x \in N \wedge x + 4 = 0\}$
   - xi) $\{x | x \in Q \wedge x^2 = 2\}$
   - xiii) $\{x | x \in \mathbb{R} \wedge x = x\}$
   - xiv) $\{x | x \in Q \wedge x = -x\}$
   - xv) $\{x | x \in \mathbb{R} \wedge x \neq x\}$
   - xvi) $\{x | x \in \mathbb{R} \wedge x \notin Q\}$

3. Which of the following sets are finite and which of these are infinite?
   - i) The set of students of your class.
   - ii) The set of all schools in Pakistan.
   - iii) The set of natural numbers between 3 and 10.
   - iv) The set of rational numbers between 3 and 10.
   - v) The set of real numbers between 0 and 1.
   - vi) The set of rationales between 0 and 1.
   - vii) The set of whole numbers between 0 and 1
   - viii) The set of all leaves of trees in Pakistan.

ix) $P(N)$ x) $P\{a, b, c\}$
xi) {1,2,3,4,...} xii) {1,2,3,....,100000000}
xiii) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$ xiv) $\{x \mid x \in \mathbb{R} \wedge x^2 = -16\}$
xv) $\{x \mid x \in Q \wedge x^2 = 5\}$ xvi) $\{x \mid x \in Q \wedge 0 \leq x \leq 1\}$

4. Write two proper subsets of each of the following sets: -
 i) $\{a, b, c\}$ ii) { 0, 1} iii) N iv) Z
 v) Q vi) $\mathbb{R}$ vii) W viii) $\{x \mid x \in Q \wedge 0 < x \leq 2\}$

5. Is there any set which has no proper sub set? If so name that set.

6. What is the difference between $\{a, b\}$ and $\{\{a, b\}\}$?

7. Which of the following sentences are true and which of them are false?
 i) { 1,2} = { 2,1} ii) $\Phi \subseteq \{\{a\}\}$ iii) $\{a\} \subseteq \{\{a\}\}$
 v) $\{a\} \in \{\{a\}\}$ vi) $a \in \{\{a\}\}$ vii) $\Phi \in \{\{a\}\}$

8. What is the number of elements of the power set of each of the following sets?
 i) { } ii) {0,1} iii) {1,2,3,4,5,6,7}
 v) {0,1,2,3,4,5,6,7} vi) $\{a, \{b, c\}\}$ vii) $\{\{a,b\},\{b,c\},\{d,e\}\}$

9. Write down the power set of each of the following sets: -
 i) {9,11 } ii) $\{+, -, \times, \div\}$ iii) $\{\Phi\}$ iv) $\{a, \{b,c\}\}$

10. Which pairs of sets are equivalent? Which of them are also equal?
 i) $\{a, b, c\}$, {1, 2, 3}
 ii) The set of the first 10 whole members, {0, 1, 2, 3,....,9}
 iii) Set of angles of a quadrilateral *ABCD*,
 set of the sides of the same quadrilateral.
 iv) Set of the sides of a hexagon *ABCDEF*,
 set of the angles of the same hexagon;
 v) {1,2,3,4,.....}, {2,4,6,8,.....} vi) {1,2,3,4,.....}, $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

viii) {5, 10, 15,.....,55555}, {5, 10, 15, 20,....... }

2.2 Operations on Sets

Just as operations of addition, subtraction etc., are performed on numbers, the operations of unions, intersection etc., are performed on sets. We are already familiar with them. A review of the main rules is given below: -

Union of two sets: The Union of two sets A and B, denoted by $A \cup B$, is the set of all elements, which belong to A or B. Symbolically;

$$A \cup B \; \{x \mid x \in A \vee x \in B\}$$

Thus if A = {1, 2, 3}, B = { 2, 3, 4, 5}, then $A \cup B$ ={1,2,3,4,5}

Notice that the elements common to A and B, namely the elements 2, 3 have been written only once in $A \cup B$ because repetition of an element of a set is not allowed to keep the elements distinct.

Intersection of two sets: The intersection of two sets A and B, denoted by $A \cap B$, is the set of all elements, which belong to both A and B. Symbolically;

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Thus for the above sets A and B, $A \cap B$ ={2,3}

Disjoint Sets: If the intersection of two sets is the empty set then the sets are said to be disjoint sets. For example; if

S_1 = The set of odd natural numbers and S_2 = The set of even natural numbers, then S_1 and S_2 are disjoint sets.

The set of arts students and the set of science students of a college are disjoint sets.

Overlapping sets: If the intersection of two sets is non-empty but neither is a subset of the other, the sets are called overlapping sets, e.g., if

L = {2,3,4,5,6} and M= {5,6,7,8,9,10}, then L and M are two overlapping sets.

Complement of a set: The complement of a set *A*, denoted by *A′* or A^c relative to the universal set *U* is the set of all elements of *U*, which do not belong to *A*.

Symbolically: $A' = \{x | x \in U \wedge x \notin A\}$

For example, if *U*=*N*, then $E' = O$ and $O' = E$

Example 1: If *U* = set of alphabets of English language,
C = set of consonants,
W = set of vowels, then $C' = W$ and $W' = C$.

Difference of two Sets: The Difference set of two sets *A* and *B* denoted by *A*–*B* consists of all the elements which belong to *A* but do not belong to *B*.

The Difference set of two sets *B* and *A* denoted by *B*-*A* consists of all the elements, which belong to *B* but do not belong to *A*.

Symbolically, $A - B = \{x | x \in A \wedge x \notin B\}$ and $B - A = \{x | x \in B \wedge x \notin A\}$

Example 2: If *A* = {1,2,3,4,5}, *B* = {4,5,6,7,8,9,10}, then
A–*B* = {1,2,3} and *B*–*A* = {6,7,8,9,10}.

Notice that $A - B \neq B - A$.

Note: In view of the definition of complement and difference set it is evident that for any set *A*, $A' = U - A$

2.3 Venn Diagrams

Venn diagrams are very useful in depicting visually the basic concepts of sets and relationships between sets. They were first used by an English logician and mathematician John Venn (1834 to 1883 A.D).

In a Venn diagram, a rectangular region represents the universal set and regions bounded by simple closed curves represent other sets, which are subsets of the universal set. For the sake of beauty these regions are generally shown as circular regions.

In the adjoining figures, the shaded circular region represents a set *A* and the remaining portion of rectangle representing the universal set *U* represents *A′* or *U* – *A*.

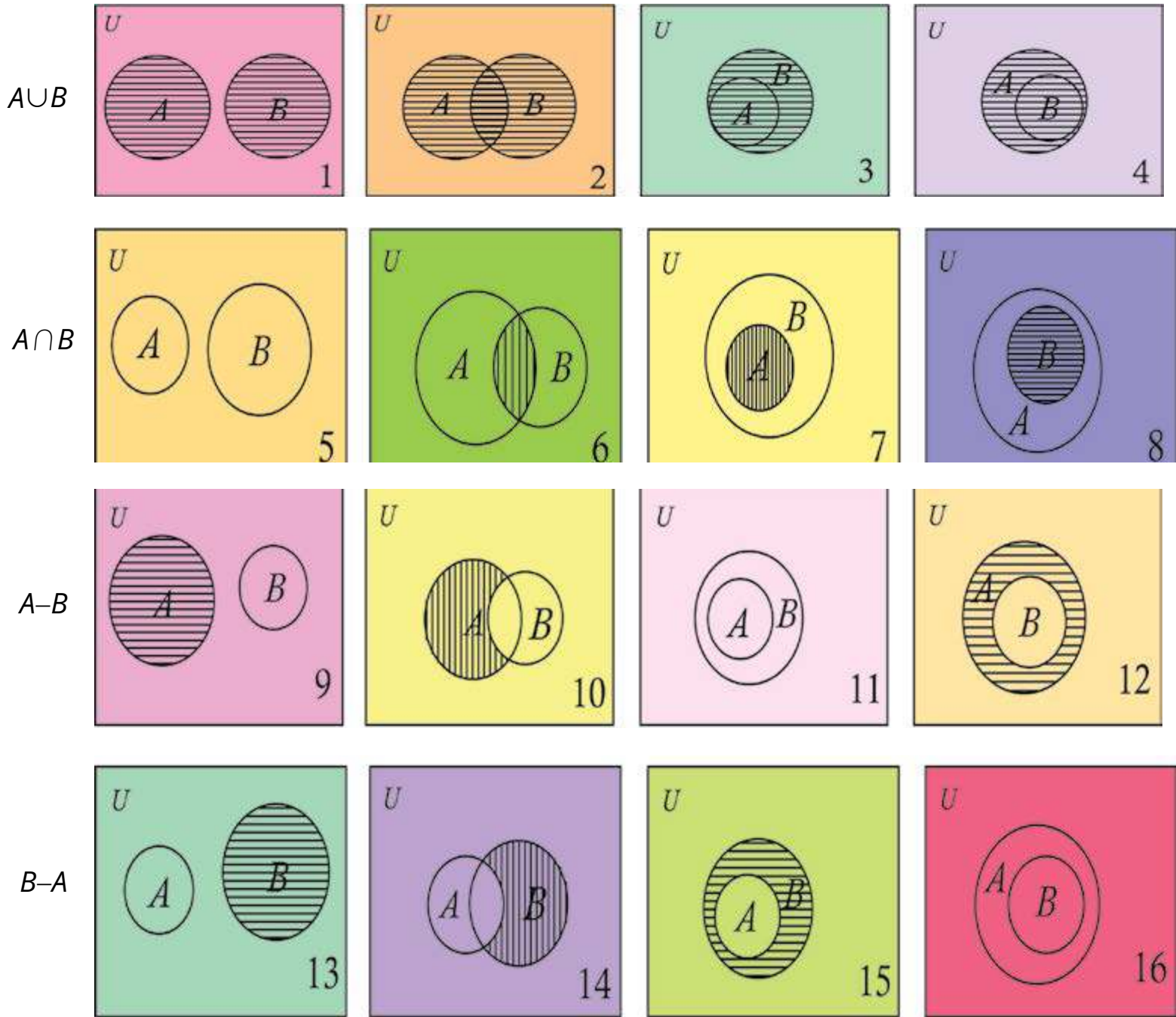
Below are given some more diagrams illustrating basic operations on two sets in different cases (lined region represents the result of the relevant operation in each case given below).

The above diagram suggests the following results: -

Fig No. Relation between A and B	Result Suggested
1. A and B disjoint sets $A \cap B = \Phi$	$A \cup B$ consists of all the elements of A and all the elements of B. Also $n(A \cup B) = n(A) + n(B)$
2. A and B are overlapping $A \cap B \neq \Phi$	$A \cup B$ contains elements which are i) in A and not in B ii) in B and not in A iii) in both A and B. Also $n(A \cup B) = n(A) + n(B) - (A \cap B)$
3. $A \subseteq B$	$A \cup B = B$; $n(A \cup B) = n(B)$
4. $B \subseteq A$	$A \cup B = A$; $n(A \cup B) = n(A)$
5. $A \cap B = \Phi$	$A \cap B = \Phi$; $n(A \cap B) = 0$
6. $A \cap B \neq \Phi$	$A \cap B$ contains the elements which are in A and B
7. $A \subseteq B$	$A \cap B = A$; $n(A \cap B) = n(A)$
8. $B \subseteq A$	$A \cap B = B$; $n(A \cap B) = n(B)$
9. A and B are disjoint sets.	$A - B = A$; $n(A - B) = n(A)$
10. A and B are overlapping	$n(A - B) = n(A) - n\ A \cap B$
11. $A \subseteq B$	$A - B = \Phi$; $n(A - B) = 0$
12. $B \subseteq A$	$A - B \neq \Phi$; $n(A - B) = n(A) - n(B)$
13. A and B are disjoint	$B - A = B$; $n(B - A) = n(B)$
14. A and B are overlapping	$n(B - A) = n(B) - n(A \cap B)$
15. $A \subseteq B$	$B - A \neq \Phi$; $n(B - A) = n(B) - n(A)$
16. $B \subseteq A$	$B - A = \Phi$; $n(B - A) = 0$

Note (1) Since the empty set contains no elements, therefore, no portion of U represents it.

(2) If in the diagrams given on preceding page we replace B by the empty set (by imagining the region representing B to vanish).

$A \cup \Phi = A$ (FromFig. 1 o r 4)
$A \cap \Phi = \Phi$ (From Fig. 5 or 8)
$A - \Phi = A$ (From Fig. 9 or 12)
$\Phi - A = \Phi$ (FromFig. 13 or 16)

Also by replacing B by A (by imagining the regions represented by A and B to coincide), we obtain the following results:

$A \cup A = A$ (From fig. 3 or 4)
$A \cap A = A$ (From fig. 7 or 8)
$A - A = \Phi$ (From fig. 12)

Again by replacing B by U, we obtain the results: -

$A \cup U = U$ (From fig. 3); $A \cap U = A$ **(From fig. 7)**
$A - U = \Phi$ (From fig. 11); $U - A = A'$ **(From fig. 15)**

(3) Venn diagrams are useful only in case of abstract sets whose elements are not specified. It is not desirable to use them for concrete sets (Although this is erroneously done even in some foreign books).

Exercise 2.2

1. Exhibit $A \cup B$ and $A \cap B$ by Venn diagrams in the following cases: -
i) $A \subseteq B$ ii) $B \subseteq A$ iii) $A \cup A'$
iv) A and B are disjoint sets. v) A and B are overlapping sets

2. Show $A - B$ and $B - A$ by Venn diagrams when: -
i) A and B are overlapping sets ii) $A \subseteq B$ iii) $B \subseteq A$

3. Under what conditions on A and B are the following statements true?
i) $A \cup B = A$ ii) $A \cup B = B$ iii) $A - B = A$
iv) $A \cap B = B$ v) $n(A \cup B) = n(A) + n(B)$ vi) $n(A \cap B) = n(A)$

vii) $A - B = A$ vii) $n(A \cap B) = 0$ ix) $A \cup B = U$
x) $A \cup B = B \cup A$ xi) $n(A \cap B) = n(B)$.   xii) $U - A = \Phi$

4. Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{2,4,6,8,10\}$, $B = \{1,2,3,4,5\}$ and $C = \{1,3,5,7,9\}$
List the members of each of the following sets: -
i) A^c ii) B^c iii) $A \cup B$ iv) $A - B$
v) $A \cap C$ vi) $A^c \cup C^c$ vii) $A^c \cup C$.   viii) U^c

5. Using the Venn diagrams, if necessary, find the single sets equal to the following: -
i) A^c ii) $A \cap U$ iii) $A \cup U$ iv) $A \cup \Phi$ v) $\Phi \cap \Phi$

6. Use Venn diagrams to verify the following: -
i) $A - B = A \cap B^c$ ii) $(A - B)^c \cap B = B$

2.4 Operations on Three Sets

If A, B and C are three given sets, operations of union and intersection can be performed on them in the following ways: -
i) $A \cup (B \cup C)$ ii) $(A \cup B) \cup C$ iii) $A \cap (B \cup C)$
iv) $(A \cap B) \cap C$ v) $A \cup (B \cap C)$ vi) $(A \cap C) \cup (B \cap C)$
vii) $(A \cup B) \cap C$ viii) $(A \cap B) \cup C$.   ix) $(A \cup C) \cap (B \cup C)$
Let $A = \{1, 2, 3\}$, $B = \{2,3,4,5\}$ and $C = \{3,4,5,6,7,8\}$
We find sets (i) to (iii) for the three sets (Find the remaining sets yourselves).
i) $B \cup C = \{2,3,4,5,6,7,8\}$, $A \cup (B \cup C) = \{1,2,3,4,5,6,7,8\}$
ii) $A \cup B = \{1,2,3,4,5\}$, $(A \cup B) \cup C = \{1,2,3,4,5,6,7,8\}$
iii) $B \cap C = \{3,4,5\}$, $A \cap (B \cap C) = \{3\}$

2.5 Properties of Union and Intersection

We now state the fundamental properties of union and intersection of two or three sets. Formal proofs of the last four are also being given.

Properties:
i) $A \cup B = B \cup A$ (Commutative property of Union)
ii) $A \cap B = B \cap A$ (Commutative property of Intersection)
iii) $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative property of Union)
iv) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative property of Intersection).
v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributivity of Union over intersection)
vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributivity of intersection over Union)

vii) $(A \cup B)' = A' \cap B'$
viii) $(A \cap B)' = A' \cup B'$
De Morgan's Laws

Proofs of De Morgan's laws and distributive laws:

i) $(A \cup B)' = A' \cap B'$
Let $x \in (A \cup B)'$
$\Rightarrow x \notin A \cup B$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \in A'$ and $x \in B'$
$\Rightarrow x \in A' \cap B'$
But x is an arbitrary member of $(A \cup B)'$ (1)
Therefore, (1) means that $(A \cup B)' \subseteq A' \cap B'$ (2)
Now suppose that $y \in A' \cap B'$
$\Rightarrow y \in A'$ and $y \in B'$
$\Rightarrow y \notin A$ and $y \notin B$
$\Rightarrow y \notin A \cup B$
$\Rightarrow y \in (A \cup B)'$
Thus $A' \cap B' \subseteq (A \cup B)'$
From (2) and (3) we conclude that (3)
$$(A \cup B)' = A' \cap B'$$

ii) $(A \cap B)' = A' \cup B'$
It may be proved similarly or deducted from (i) by complementation

iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Let $x \in A \cup (B \cap C)$
$\Rightarrow x \in A$ or $x \in B \cap C$
$\Rightarrow$ If $x \in A$ it must belong to $A \cup B$ and $x \in A \cup C$

$\Rightarrow \ x \in (A \cup B) \cap (A \cup C)$

Also if $x \in B \cap C$, then $x \in B$ and $x \in C$. (1)

$\Rightarrow \ x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow \ x \in (A \cup B) \cap (A \cup C)$

Thus $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (2)

Conversely, suppose that

$y \in (A \cup B) \cap (A \cup C)$

There are two cases to consider: -

$y \in A, \ y \notin A$

In the first case y $\in A \cup (B \cap C)$

If $y \notin A$, it must belong to B as well as C

i.e., y $\in (B \cap C)$

$\therefore y \in A \cup (B \cap C)$

So in either case

y$\in (A \cup B) \cap (A \cup C) \Rightarrow$ y$\in A \cup (B \cap C)$

thus $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ (3)

From (2) and (3) it follows that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

It may be proved similarly or deducted from (iii) by complementation

Verification of the properties:

Example 1: Let $A = \{1,2,3\}$, $B = \{2,3,4,5\}$ and $C = \{3,4,5,6,7,8\}$

i) $A \cup B = \{1,2\ 3\} \cup \{2,3,4,5\}$ $\quad B \cup A = \{2,3,4,5\} \cup \{1,2,3\}$

$= \{1,2,3,4,5\}$ $\quad = \{2,3,4,5,1\}$

$\therefore A \cup B = B \cup A$

ii) $A \cap B = \{1,2,3\} \cap \{2,3,4,5\}$ $\quad B \cap A = \{2,3,4,5\} \cap \{1,2,3\}$

$= \{2,3\}$ $\quad = \{2,3\}$

$\therefore A \cap B = B \cap A$

(iii) and (iv) Verify yourselves.

(v) $A \cup (B \cap C) = \{1,2,3\} \cup (\{2,3,4,5\} \cap \{3,4,5,6,7,8)$

$= \{1,2,3\} \cup \{3,4,5\}$

$= \{1,2,3,4,5\}$ (1)

$(A \cup B) \cap (A \cup C) = (\{1,2,3\} \cup \{2,3,4,5\}) \cap (\{1,2,3\} \cup \{3,4,5,6,7,8\})$

$= \{1,2,3,4,5\} \cap \{1,2,3,4,5,6,7,8\}$

$= \{1,2,3,4,5\}$ (2)

From (1) and (2),

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

vi) Verify yourselves.

vii) Let the universal set be $U = \{1,2,3,4,5,6,7,8,9,10\}$

$A \cup B = \{1,2,3\} \cup \{2,3,4,5\} = \{1,2,3,4,5\}$

$(A \cup B)' = \{6,7,8,9,10\}$ (1)

$A' = U - A = (4,5,6,7,8,9,10)$

$B' = U - B = \{1,6,7,8,9,10\}$

$A' \cap B' = (4,5,6,7,8,9,10\} \cap \{1,6,7,8,9,10\}$

$= \{6,7,8,9,10\}$ (2)

From (1) and (2),

$(A \cup B)' = A' \cap B'$

viii) Verify yourselves.

Verification of the properties with the help of Venn diagrams.

i) and (ii): Verification is very simple, therefore, do it yourselves,

iii): In fig. (1) set A is represented by vertically lined region and $B \cup C$ is represented by horizontally lined region. The set $A \cup (B \cup C)$ is represented by the region which is lined either in one or both ways.

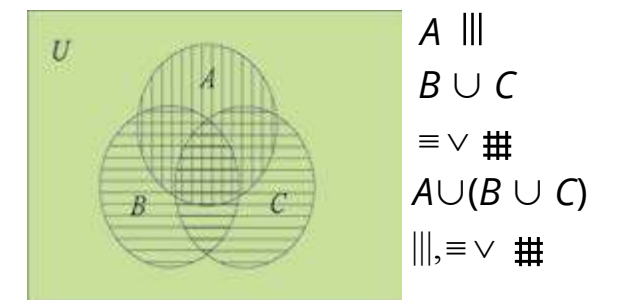



Fig (1)

In figure(2) $A \cup B$ is represented by horizontally lined region and C by vertically lined region. $(A \cup B) \cup C$ is represented by the region which is lined in either one or both ways.

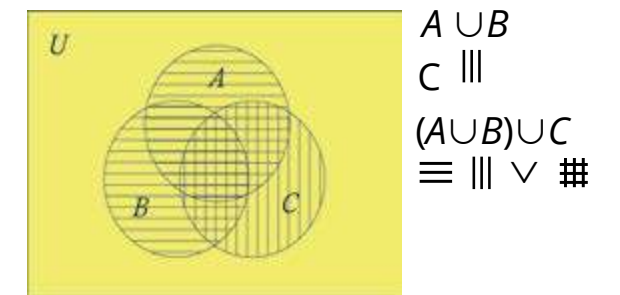
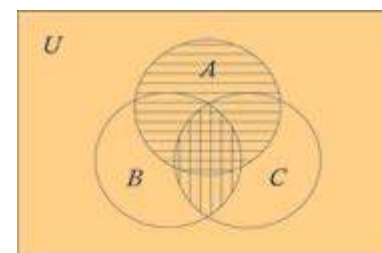



Fig (2)

From fig (1) and (2) we can see that

$A \cup (B \cup C) = (A \cup B) \cup C$

(iv) In fig (3) doubly lined region represents.

$A \cap (B \cap C)$

$A \equiv$
$B \cap C$ |||
$A \cap (B \cap C$

Fig (3)

In fig (4) doubly lined region represents

$(A \cap B) \cap C$.

Since in fig (3) and (4) these regions are the same therefore,

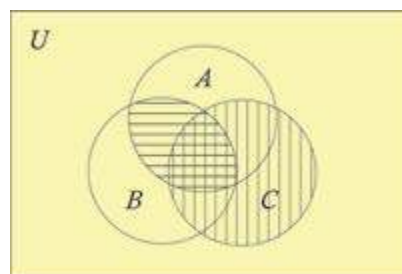
$A \cap (B \cap C) = (A \cap B) \cap C$.

$A \cap B \equiv$
C |||
$(A \cap B) \cap C$
#

Fig (4)

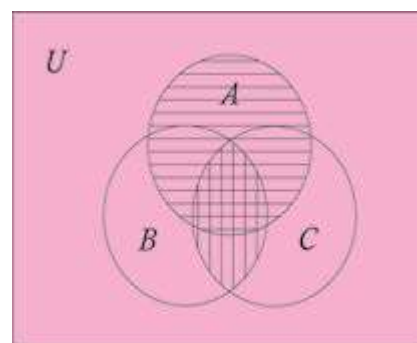
(v) in fig. (5) $A \cup (B \cap C)$ is represented by the region which is lined horizontally or vertically or both ways.

$A \equiv$
$B \cap C$ |||
$A \cup (B \cap C)$
≡ , ||| ∨ #

Fig (5)

In fig. (6) $(A \cup B) \cap (A \cup C)$ is represented by the doubly lined region. Since the two region in fig (5) and (6) are the same, therefore

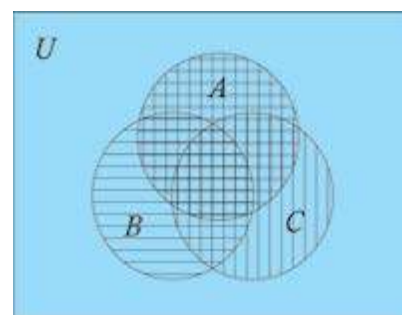



Fig (6)

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(vi) Verify yourselves.

(vii) In fig (7) $(A \cup B)'$ is represented by vertically lined region.

$A \cup B \equiv$
$(A \cup B)'$ |||

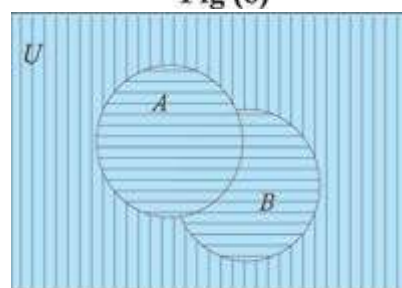



Fig (7)

In fig. (8) doubly lined region represents.

$A' \cap B'$.

The two regions in fig (7). And (8) are the same, therefore, $(A \cup B)' = A' \cap B'$

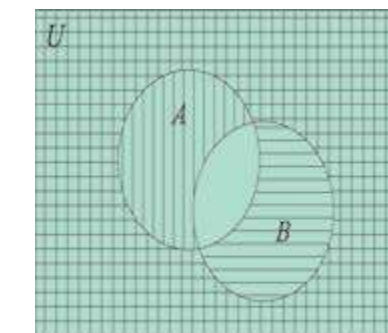



Fig (8)

(viii) Verify yourselves.

Note: In all the above Venn diagrams only overlapping sets have been considered. Verification in other cases can also be effected similarly. Detail of verification may be written by yourselves.

Exercise 2.3

1. Verify the commutative properties of union and intersection for the following pairs of sets: -

i) $A = \{1,2,3,4,5\}$, $B = \{4,6,8,10\}$ ii) N, Z

iii) $A = \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$, $B = \mathbb{R}$.

2. Verify the properties for the sets A, B and C given below: -

i) Associativity of Union ii) Associativity of intersection.

iii) Distributivity of Union over intersection.

iv) Distributivity of intersection over union.

a) $A = \{1,2,3,4\}$, $B = \{3,4,5,6,7,8\}$, $C = \{5,6,7,9,10\}$

b) $A = \Phi$, $B = \{0\}$, $C = \{0,1,2\}$

c) N, Z, Q

3. Verify De Morgan's Laws for the following sets:

$U = \{1,2,3, ..., 20\}$, $A = \{2,4,6,...., 20\}$ and $B = \{1,3,5, ...,19\}$.

4. Let U = The set of the English alphabet

$A = \{x \mid x$ is a vowel$\}$, $B = \{y \mid y$ is a consonant$\}$,

Verify De Morgan's Laws for these sets.

5. With the help of Venn diagrams, verify the two distributive properties in the following

cases w.r.t union and intersection.

i) $A \subseteq B$, $A \cap C = \Phi$ and B and C are overlapping.

ii) A and B are overlapping, B and C are overlapping but A and C are disjoint.

6. Taking any set, say $A = \{1,2,3,4,5\}$ verify the following: -

i) $A \cup \Phi = A$ ii) $A \cup A = A$ iii) $A \cap A = A$

7. If $U = \{1,2,3,4,5,...., 20\}$ and $A = \{1,3,5, ..., 19\}$, verify the following:-

i) $A \cup A' = U$ ii) $A \cap U = A$ iii) $A \cap A' = \Phi$

8. From suitable properties of union and intersection deduce the following results:

i) $A \cap (A \cup B) = A \cup (A \cap B)$ ii) $A \cup (A \cap B) = A \cap (A \cup B)$.

9. Using venn diagrams, verify the following results.

i) $A \cap B' = A$ iff $A \cap B = \Phi$ ii) $(A - B) \cup B = A \cup B$.

iii) $(A - B) \cap B = \Phi$ iv) $A \cup B = A \cup (A' \cap B)$.

2.6 Inductive and Deductive Logic

In daily life we often draw general conclusions from a limited number of observations or experiences. A person gets penicillin injection once or twice and experiences reaction soon afterwards. He generalises that he is allergic to penicillin. We generally form opinions about others on the basis of a few contacts only. This way of drawing conclusions is called **induction**.

Inductive reasoning is useful in natural sciences where we have to depend upon repeated experiments or observations. In fact greater part of our knowledge is based on induction.

On many occasions we have to adopt the opposite course. We have to draw conclusions from accepted or well-known facts. We often consult lawyers or doctors on the basis of their good reputation. This way of reasoning i.e., drawing conclusions from premises believed to be true, is called **deduction**. One usual example of deduction is: All men are mortal. We are men. Therefore, we are also mortal.

Deduction is much used in higher mathematics. In teaching elementary mathematics we generally resort to the inductive method. For instance the following sequences can be continued, inductively, to as many terms as we like:

i) 2,4,6,... ii) 1,4,9,... iii) 1,-1,2,-2,3,-3,...

iv) 1,4,7,... v) $\frac{1}{3}, \frac{1}{12}, \frac{1}{36}, ...$ vi) $\frac{1}{10}, \frac{2}{100}, \frac{4}{1000}, ...$

As already remarked, in higher mathematics we use the deductive method. To start with we accept a few statements (called postulates) as true without proof and draw as many conclusions from them as possible.

Basic principles of deductive logic were laid down by Greek philosopher, Aristotle. The illustrious mathematician Euclid used the deductive method while writing his 13 books of geometry, called Elements. Toward the end of the 17th century the eminent German mathematician, Leibniz, symbolized deduction. Due to this device deductive method became far more useful and easier to apply.

2.6.1 Aristotelian and non-Aristotelian logics

For reasoning we have to use *propositions*. A daclarative statement which may be true or false but not both is called a proposition. According to Aristotle there could be only two possibilities - a proposition could be either true or false and there could not be any third possibility. This is correct so far as mathematics and other exact sciences are concerned. For instance, the statement $a = b$ can be either true or false. Similarly, any physical or chemical theory can be either true or false. However, in statistical or social sciences it is sometimes not possible to divide all statements into two mutually exclusive classes. Some statements may be, for instance, undecided.

Deductive logic in which every statement is regarded as true or false and there is no other possibility, is called Aristotlian Logic. Logic in which there is scope for a third or fourth possibility is called non-Aristotelian. we shall be concerned at this stage with Aristotelian logic only.

2.6.2 Symbolic logic

For the sake of brevity propositions will be denoted by the letters p , q etc. We give a

brief list of the other symbols which will be used.

Symbol	How to be read	Symbolic expression	How to be read
$\sim$	not	$\sim p$	Not p, negation of p
$\wedge$	and	$p \wedge q$	p and q
$\vee$	or	$p \vee q$	p or q
$\rightarrow$	If... then, implies	$p \rightarrow q$	If p then q p implies q
$\leftrightarrow$	Is equivalent to, if and only if	$p \leftrightarrow q$	p if and only if q p is equivalent to q

Explanation of the use of the Symbols:

1) **Negation:** If p is any proposition its negation is denoted by $\sim p$, read 'not p'. It follows from this definition that if p is true, $\sim p$ is false and if p is false, $\sim p$ is true. The adjoining table, called truth table, gives the possible truth- values of p and $\sim p$.

p	$\sim p$
T	F
F	T

Table (1)

2) **Conjunction** of two statements p and q is denoted symbolically as $p \wedge q$ (p and q). A conjunction is considered to be true only if both its components are true. So the truth table of $p \wedge q$ is table (2).

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table (2)

Example 1:

i) Lahore is the capital of the Punjab and Quetta is the capital of Balochistan.

ii) $4 < 5 \wedge 8 < 10$

iii) $4 < 5 \wedge 8 > 10$

iv) $2 + 2 = 3 \wedge 6 + 6 = 10$

Clearly conjunctions (i) and (ii) are true whereas (iii) and (iv) are false.

3) **Disjunction** of p and q is p or q. It is symbolically written $p \vee q$. The disjunction $p \vee q$ is considered to be true when at least one of the components p and q is true. It is false when both of them are false. Table (3) is the truth table.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table (3)

Example 2:

i) 10 is a positive integer or π is a rational number. Find truth value of this disjunction.

Solution: Since the first component is true, the disjunction is true.

ii) A triangle can have two right angles or Lahore is the capital of Sind.

Solution: Both the components being false, the composite proposition is false.

2.7 Implication or conditional

A compound statement of the form if p then q, also written ***p* implies *q***, is called a **conditional** or an **implication**, p is called the **antecedent** or **hypothesis** and q is called the **consequen**t or the **conclusion**.

A conditional is regarded as false only when the antecedent is true and consequent is false. In all other cases it is considered to be true. Its truth table is, therefore, of the adjoining form.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table (4)

Entries in the first two rows are quite in consonance with common sense but the entries of the last two rows seem to be against common sense. According to the third row the conditional

If p then q

is true when p is false and q is true and the compound proposition is true (according to the fourth row of the table) even when both its components are false. We attempt to clear the position with the help of an example. Consider the conditional

If a person A lives at Lahore, then he lives in Pakistan.

If the antecedent is false i.e., A does not live in Lahore, all the same he may be living in Pakistan. We have no reason to say that he does not live in Pakistan.
We cannot, therefore, say that the conditional is false. So we must regard it as true. It must be remembered that we are discussing a problem of Aristotlian logic in which every proposition must be either true or false and there is no third possibility. In the case under discussion there being no reason to regard the proposition as false, it has to be regarded as true. Similarly, when both the antecedent and consequent of the conditional under consideration are false, there is no justification for quarrelling with the proposition. Consider another example.

A certain player, *Z*, claims that if he is appointed captain, the team will win the tournament. There are four possibilities: -

i) *Z* is appointed captain and the team wins the tournament. *Z*'s claim is true.

ii) *Z* is appointed captain but the team loses the tournament. *Z*'s claim is falsified.

iii) *Z* is not appointed captain but the team all the same wins the tournament.There is no reason to falsify *Z*'s claim.

iv) *Z* is not appointed captain and the team loses the tournament. Evidently, blame cannot be put on *Z*.

It is worth noticing that emphasis is on the conjunction if occurring in the beginning of the ancedent of the conditional. If condition stated in the antecedent is not satisfied we should regard the proposition as true without caring whether the consequent is true or false.

For another view of the matter we revert to the example about a Lahorite:

'If a person A lives at Lahore, then he lives in Pakistan'.

p: A person A lives at Lahore.

q: He lives in Pakistan

When we say that this proposition is true we mean that in this case it is not possible that '*A* lives at Lahore' is true and that '*A* does not live in Pakistan' is also true, that is $p \to q$ and $\sim(p \wedge \sim q)$ are both simultaneously true. Now the truth table of $\sim(p \wedge \sim q)$ is shown below:

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Table (5)

Looking at the last column of this table we find that truth values of the compound proposition $\sim(p \wedge \sim q)$ are the same as those adopted by us for the conditional $p \to q$. This shows that the two propositions $p \to q$ and $\sim(p \wedge \sim q)$ are logically equivalent. Therefore, the truth values adopted by us for the conditional are correct.

2.7.1 Biconditional : $p \leftrightarrow q$

The proposition $p \to q \wedge q \to p$ i s shortly written $p \leftrightarrow q$ and is called the **biconditional** or **equivalence**. It is read ***p*** iff ***q*** (iff stands for "if and only if ')

We draw up its truth table.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Table (6)

From the table it appears that $p \leftrightarrow q$ is true only when both p and q are true or both p and q are false.

2.7.2 Conditionals related with a given conditional.

Let $p \to q$ be a given conditional. Then

i) $q \to p$ is called the **converse** of $p \to q$;

ii) $\sim p \to \sim q$ is called the **inverse** of $p \to q$;

iii) $\sim q \to \sim p$ is called the **contrapositive** of $p \to q$.

To compare the truth values of these new conditionals with those of $p \to q$ we draw up their joint table.

				Given conditional	Converse	Inverse	Contrapositive
p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	F	T	T	T	T

Table (7)

From the table it appears that

i) Any conditional and its contrapositive are equivalent therefore any theorem may be proved by proving its contrapositive.

ii) The converse and inverse are equivalent to each other.

Example 3: Prove that in any universe the empty set ϕ is a subset of any set A.

First Proof: Let U be the universal set consider the conditional:

$$\forall x \in U,\ x \in \phi \rightarrow x \in A \qquad (1)$$

The antecedent of this conditional is false because no $x \in U$, is a member of ϕ.
Hence the conditional is true.

Second proof: (By contrapositive)
The contrapositive of conditional (1) is

$$\forall x \in U,\ x \notin A \rightarrow x \notin \phi \qquad (2)$$

The consequent of this conditional is true. Therefore, the conditional is true.
Hence the result.

Example 4: Construct the truth table ot $[(p \rightarrow q) \wedge p \rightarrow q]$

Solution : Desired truth table is given below: -

p	p	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p \rightarrow q]$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Table (8)

2.7.3 Tautologies

i) A statement which is true for all the possible values of the variables involved in it is

called a tautology, for example, $p \rightarrow q \leftrightarrow (\sim q \rightarrow \sim p)$ is a *tautology*.(are already verified by a truth table).

ii) A statement which is always false is called an **absurdity** or a contradiction e.g., $p \rightarrow \sim p$

iii) A statement which can be true or false depending upon the truth values of the variables involved in it is called a **contingency** e.g., $(p \rightarrow q) \wedge (p \vee q)$ is a contingency.
(You can verify it by constructing its truth table).

2.7.4 Quantifiers

The words or symbols which convey the idea of quantity or number are called quantifiers.
In mathematics two types of quantifiers are generally used.

i) **Universal quantifier** meaning for all
Symbol used : $\forall$

ii) **Existential quantifier:** There exist (some or few, at least one) symbol used: $\exists$

Example 5:

i) $\forall x \in A$, $p(x)$ is true.
(To be read : For all x belonging to A the statement $p(x)$ is true).

ii) $\exists x \in A \ni p(x)$ is true.
(To be read : There exists x belonging to A such that statement $p(x)$ is true).

The symbol $\ni$ stands for such that

Exercise 2.4

1. Write the converse, inverse and contrapositive of the following conditionals: -
   i) $\sim p \rightarrow q$ ii) $q \rightarrow p$ iii) $\sim p \rightarrow \sim q$ iv) $\sim q \rightarrow \sim p$

2. Construct truth tables for the following statements: -
   i) $(p \rightarrow \sim p) \vee (p \rightarrow q)$ ii) $(p \wedge \sim p) \rightarrow q$

iii) $\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$

3. Show that each of the following statements is a tautology: -
   i) $(p \wedge q) \rightarrow p$ ii) $p \rightarrow (p \vee q)$
   iii) $\sim(p \rightarrow q) \rightarrow p$ iv) $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$

4. Determine whether each of the following is a tautology, a contingency or an absurdity: -
   i) $p \wedge \sim p$ ii) $p \rightarrow (q \rightarrow p)$ iii) $q \vee (\sim q \vee p)$
5. Prove that $p \vee (\sim p \wedge \sim q) \vee (p \wedge q) = p \vee (\sim p \wedge \sim q)$

2.8 Truth Sets, A link between Set Theory and Logic.

Logical propositions p, q etc., are formulae expressed in terms of some variables. For the sake of simplicity and convenience we may assume that they are all expressed in terms of a single variable x where x is a real variable. Thus $p = p(x)$ where, $x \in \mathbb{R}$. All those values of x which make the formula $p(x)$ true form a set, say P. Then P is the truth set of p. Similarly, the **truth set**, Q, of q may be defined. We can extend this notion and apply it in other cases.
i) **Truth set of ~*p*:** Truth set of $\sim p$ will evidently consist of those values of the variable for which p is false i.e., they will be members of P' , the complement of P.
ii) **$p \vee q$:** Truth set of $p \vee q = p(x) \vee q(x)$ consists of those values of the variable for which $p(x)$ is true or $q(x)$ is true or both $p(x)$ and $q(x)$ are true.
Therefore, truth set of $p \vee q$ will be:

$$P \cup Q = \{x \mid p(x) \text{ is true or } q(x) \text{ is true}\}$$

iii) **$p \wedge q$:** Truth set of $p(x) \wedge q(x)$ will consist of those values of the variable for which both $p(x)$ and $q(x)$ are true. Evidently truth set of

$$p \wedge q = P \cap Q$$
$$= \{x \mid p(x) \text{ is true } \wedge q(x) \text{ is true}\}$$

iv) **$p \rightarrow q$:** We know that $p \rightarrow q$ is equivalent to $\sim p \vee q$ therefore truth set of $p \rightarrow q$ will be $P' \cup Q$

v) **$p \leftrightarrow q$:** We know that $p \leftrightarrow q$ means that p and q are simultaneously true or false. Therefore, in this case truth sets of p and q will be the same i.e.,

$$P = Q$$

Note: **(1)** Evidently truth set of a tautology is the relevant universal set and that of an absurdity is the empty set Φ.
(2) With the help of the above results we can express any logical formula in set-theoretic form and vice versa.
We will illustrate this fact with the help of a solved example.

Example 1: Give logical proofs of the following theorems: -
(A, B and C are any sets)
i) $(A \cup B)' = A' \cap B'$ ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution: i) The corresponding formula of logic is

$$\sim(p \vee q) = \sim p \wedge \sim q \quad (1)$$

We construct truth table of the two sides.

p	p	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

The last two columns of the table establish the equality of the two sides of eq.(1)
(ii) Logical form of the theorem is
$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
We construct the table for the two sides of this equation

1	2	3	4	⑤	6	7	⑧
p	p	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Comparison of the entries of columns ⑤ and ⑧ is sufficient to establish the desired result.

Exercise 2.5

Convert the following theorems to logical form and prove them by constructing truth tables: -

1. $(A \cap B)' = A' \cup B'$
2. $(A \cup B) \cup C = A \cup (B \cup C)$
3. $(A \cap B) \cap C = A \cap (B \cap C)$
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2.9 Relations

In every-day use *relation* means an abstract type of connection between two persons or objects , for instance, (Teacher, Pupil), (Mother, Son), (Husband, Wife), (Brother, Sister), (Friend, Friend), (House, Owner). In mathematics also some operations determine relationship between two numbers, for example: -

> : (5 , 4); square: (25, 5); Square root: (2,4); Equal: (2 $\times$ 2, 4).

Technically a *relation* is a set of ordered pairs whose elements are ordered pairs of related numbers or objects. The relationship between the components of an ordered pair may or may not be mentioned.

i) Let A and B be two non-empty sets, then any subset of the Cartesian product $A \times B$ is called a **binary relation**, or simply a **relation**, from A to B. Ordinarily a relation will be denoted by the letter r.

ii) The set of the first elements of the ordered pairs forming a relation is called its **domain**.

iii) The set of the second elements of the ordered pairs forming a relation is called its **range**.

iv) If A is a non-empty set, any subset of $A \times A$ is called a **relation in *A***. Some authors call it a **relation on *A***.

Example 1: Let c_1, c_2, c_3 be three children and m_1, m_2 be two men such that father of both c_1, c_2 is m_1 and father of c_3 is m_2. Find the relation {(child, father)}

Solution: C = Set of children = $\{c_1, c_2, c_3\}$ and F = set of fathers = $\{m_1, m_2\}$

$C \times F = \{(c_1, m_1), (c_1, m_2), (c_2, m_1), (c_2, m_2), (c_3, m_1), (c_3, m_2)\}$

r = set of ordered pairs (*child, father*).

$= \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$

Dom $r = (c_1, c_2, c_3\}$, Ran $r = \{m_1, m_2\}$

The relation is shown diagrammatically in fig. (2.29).

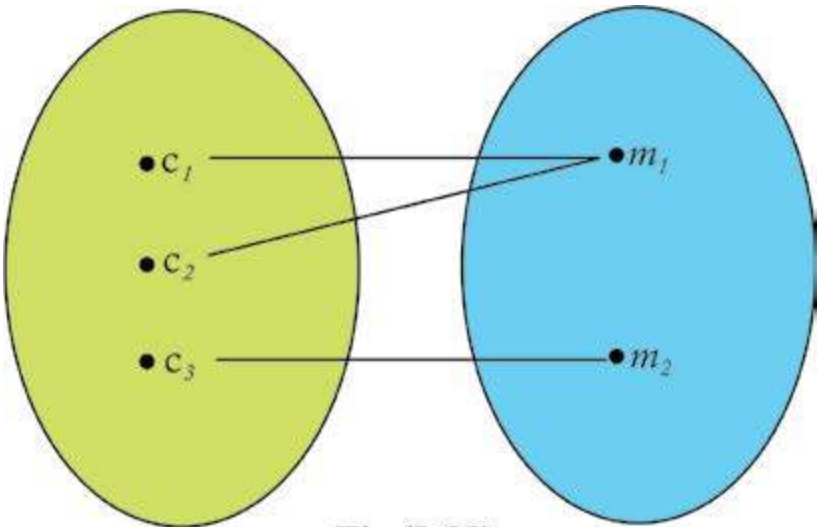



Fig (2.29)

Example 2: Let $A = \{1, 2, 3\}$. Determine the relation r such that xry iff $x < y$.

Solution: $A \times A = \{(1, 1),(1, 2),(1, 3),(2, 1), (2, 2),(2, 3),(3, 1),(3, 2),(3, 3)\}$

Clearly, required relation is:

$r = \{(1, 2), (1, 3), (2, 3)\}$, *Dom* $r = \{1, 2\}$, Ran $r = \{2, 3\}$

Example 3: Let $A = \mathbb{R}$, the set of all real numbers.

Determine the relation r such that $x \, r \, y$ iff $y = x + 1$

Solution: $A \times A = \mathbb{R} \times \mathbb{R}$

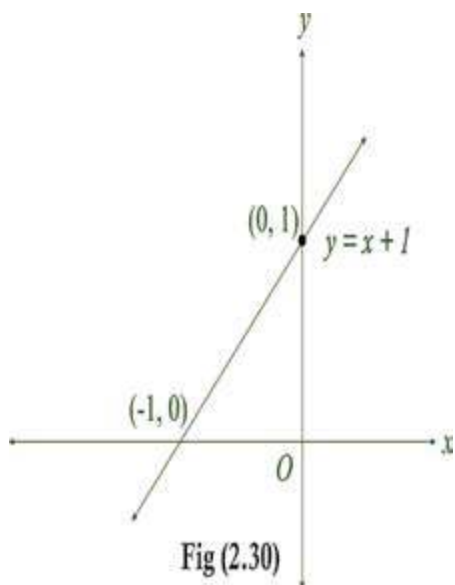
$r = \{ (x, y) | \ y = x + 1\}$

When $x = 0$, $y = 1$

$x = -1$, $y = 0$,

r is represented by the line passing through the points (0,1), (– 1,0).

Fig (2.30)

Some more points belonging to r are:
{(1, 2), (2, 3), (3, 4), (–2, –1), (–3 ,–2),(–4,–3)}
Clearly, Dom $r = \mathbb{R}$, and Ran $r = \mathbb{R}$

2.10 Functions

A very important special type of relation is a function defined as below: -
Let A and B be two non-empty sets such that:

i) f is a relation from A t o B that is , f is a subset of $A \times B$
ii) Dom $f = A$
iii) First element of no two pairs of f are equal, then f is said to be a function from A to B.

The function f is also written as:

$$f : A \rightarrow B$$

which is read: f is a function from A t o B .
If (x, y) in an element of f when regarded as a set of ordered pairs,
we write $y = f(x)$. y is called the value of f for x or image of x under f.
In example 1 discussed above

i) r is a subset of $C \times F$
ii) Dom $r = \{c_1, c_2, c_3\} = C$;
iii) First elements of no two related pairs of r are the same.

Therefore, r is a function from C to F.
In Example 2 discussed above

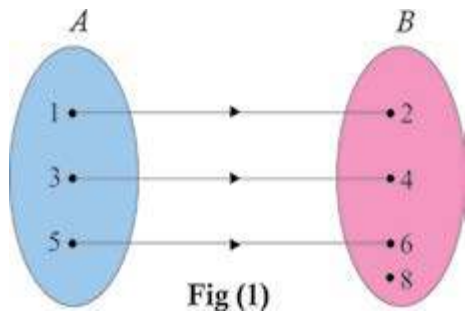
i) r is a subset of $A \times A$;
ii) Dom $r \neq A$

Therefore, the relation in this case is not a function.
In example 3 discussed above

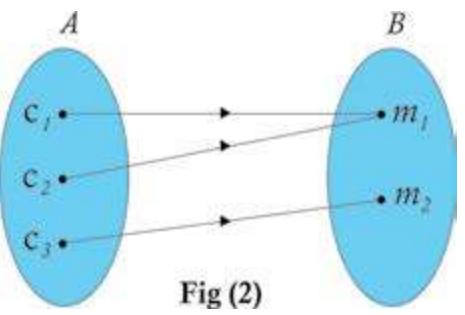
i) r is a subset of $\mathbb{R}$

ii) Dom $r = \mathbb{R}$
iii) Clearly first elements of no two ordered pairs of r can be equal. Therefore, in this case r is a function.

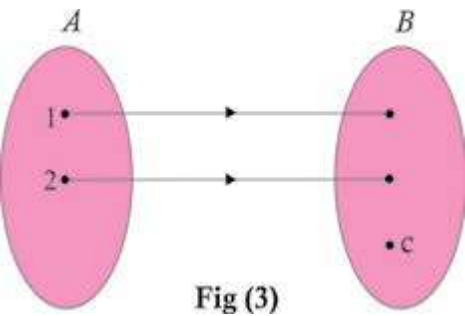
i) Into Function: If a function $f : A \rightarrow B$ is such that Ran $f \subset B$ i.e., Ran $f \neq$ B, then f is said to be a function from A **into** B. In fig.(1) f is clearly a function. But Ran $f \neq$ B. Therefore, f is a function from A into B.

Fig (1)

$f = \{(1,2), (3,4), (5,6)\}$

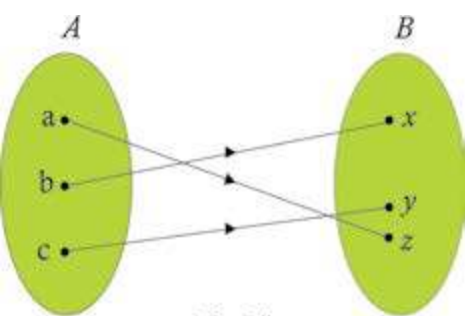
ii) Onto (Surjective) function: If a function $f : A \rightarrow B$ is such that Ran $f = B$ i.e., every element of B is the image of some elements of A, then f is called an **onto** function or a surjective function.

Fig (2)

$f = \{(c_1, m_1),(c_2, m_1),(c_3, m_2)\}$

iii) (1-1) and into (Injective) function: If a function f from A into B is such that second elements of no two of its ordered pairs are equal, then it is called an injective (1 - 1, and into) function. The function shown in fig (3) is such a function.

Fig (3)

$f = \{(1, a)(2, b)\}$

iv) (1 –1) and Onto function (bijective function). If f is a function from A onto B such that second elements of no two of its ordered pairs are the same, then f is said to be (1 – 1) function from A onto B.
Such a function is also called a (1 – 1) correspondence between A and B. It is also called a bijective function.
Fig(4) shows a (1-1) correspondence between the sets A and B.

Fig (4)

$f = \{(a, z),(b, x),(c, y)\}$

(a, z), (b, x) and (c, y) are the pairs of corresponding elements i.e., in this case $f = \{(a, z), (b, x), (c, y)\}$ which is a bijective function or (1 - 1) correspondence between the sets A and B.

Set - Builder Notation for a function: We know that set-builder notation is more suitable for infinite sets. So is the case in respect of a function comprising infinite number of ordered pairs. Consider for instance, the function

$f = \{ (1,1), (2,4), (3, 9), (4, 16),...\}$

Dom $f = (1, 2, 3, 4, ...\}$.and Ran $f = \{1,4,9, 16, ...\}$

This function may be written as: $f = \{(x, y) \mid y = x^2, x \in N\}$

For the sake of brevity this function may be written as:

f = function defined by the equation $y = x^2$, $x \in N$

Or, to be still more brief: The function x^2, $x \in N$

In algebra and Calculus the domain of most functions is $\mathcal{R}$ and if evident from the context it is, generally, omitted.

2.10.1 Linear and Quadratic Functions

The function $\{(x, y) \mid y = mx + c\}$ is called a **linear function**, because its graph (geometric representation) is a straight line. Detailed study of a straight line will be undertaken in the next class. For the present it is sufficient to know that an equationof the form

$y = mx + c$ or $ax + by + c = 0$ represents a straight line . This can be easily verified by drawing graphs of a few linear equations with numerical coefficients. The function $\{(x, y) \mid y = ax^2 + bx + c\}$ is called a **quadratic function** because it is defined by a quadratic (second degree) equation in x, y.

Example 4: Give rough sketch of the functions

i) $\{(x, y) \mid 3x + y = 2\}$ ii) $\{(x, y) \mid y = \frac{1}{2}x^2\}$

Solution:

i) The equation defining the function is $3x + y = 2$

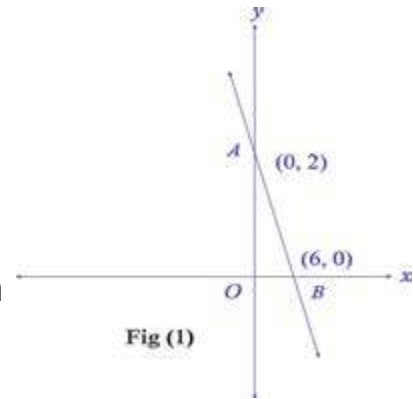
$\Rightarrow y = -3x + 2$

We know that this equation, being linear, represents a straight line. Therefore, for drawing its sketch or graph only two of its points are sufficient.

When $x = 0, y = 2$,

When $y = 0, x = \frac{2}{3} = 0.6$ nearly. So two points on the line are A (0, 2) and B = (0.6, 0).

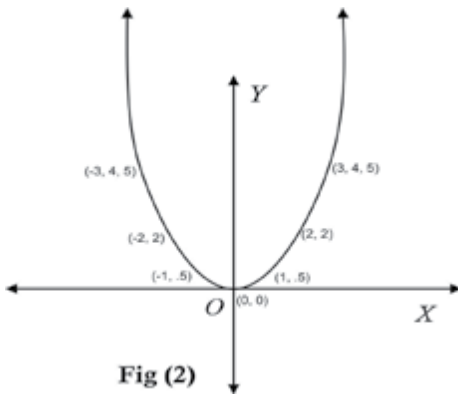
Joining A and B and producing $\overline{AB}$ in both directions, we obtain the line AB i.e., graph of the given function.

Fig (1)

ii) The equation defining the function is $y = \frac{1}{2}x^2$.

Corresponding to the values 0, ± 1, ±2, ±3 ... of x, values of y are 0, .5, 2, 4.5, ...

We plot the points (0, 0), (±1, .5), (±2, 2), (±3, 4.5), ...

Joining them by means of a smooth curve and extending it upwards we get the required graph. We notice that:

Fig (2)

i) The entire graph lies above the x-axis.

ii) Two equal and opposite values of x correspond to every value of y (but not vice versa).

iii) As x increases (numerically) y increases and there is no end to their ncrease. Thus the graph goes infinitely upwards. Such a curve is called a parabola. The students will learn more about it in the next class.

2.11 Inverse of a function

If a relation or a function is given in the tabular form i.e., as a set of ordered pairs, its inverse is obtained by interchanging the components of each ordered pair. The inverse of r and f are denoted r^{-1} and f^{-1} respectively.

If r or f are given in set-builder notation the inverse of each is obtained by interchanging x and y in the defining equation. The inverse of a function may or may not be a function.

The inverse of the linear function

$\{(x, y) \mid y = mx + c\}$ is $\{(x, y) \mid x = my + c\}$ which is also a linear function. Briefly, we may say that the **inverse of a line is a line.**

The line $y = x$ is clearly self-inverse. The function defined by this equation i.e., the function $\{(x, y) \mid y = x\}$ is called the identity function.

Example 6: Find the inverse of

i) $\{(1, 1\}, (2,4), (3,9), (4, 16), \ldots x \in Z^+\}$,

ii) $\{(x, y) \mid y = 2x + 3, x \in \mathbb{R}\}$ iii) $\{(x, y) \mid x^2 + y^2 = a^2\}$.

Tell which of these are functions.

Solution:

i) The inverse is:

$\{(2,1), (4, 2), (9, 3), (16, 4) \ldots\}$.

This is also a function.

Note: Remember that the equation

$y = \sqrt{x}, x \geq 0$

defines a function but the equation $y^2 = x, x \geq 0$ does not define a function.

The function defined by the equation

$y = \sqrt{x}, x \geq 0$

is called the **square root function.**

The equation $y^2 = x \Rightarrow y = \pm\sqrt{x}$

Therefore, the equation $y^2 = x$ $(x \geq 0)$ may be regarded as defining the union of the functions defined by

$y = \sqrt{x}, x \geq 0$ and $y = -\sqrt{x}, x \geq 0$.

ii) The given function is a **linear function**. Its inverse is:

$\{(x, y) \mid x = 2y + 3\}$

which is also a linear function.

Points (0, 3), (–1.5, 0) lie on the given line and points (3, 0), (0, –1.5) lie on its inverse. (Draw the graphs yourselves).

The lines *l*, *i'* are symmetric with respect to the line $y = x$. This quality of symmetry is true not only about a linear *n* function and its inverse but is also true about any function of a higher degree and its inverse (why?).

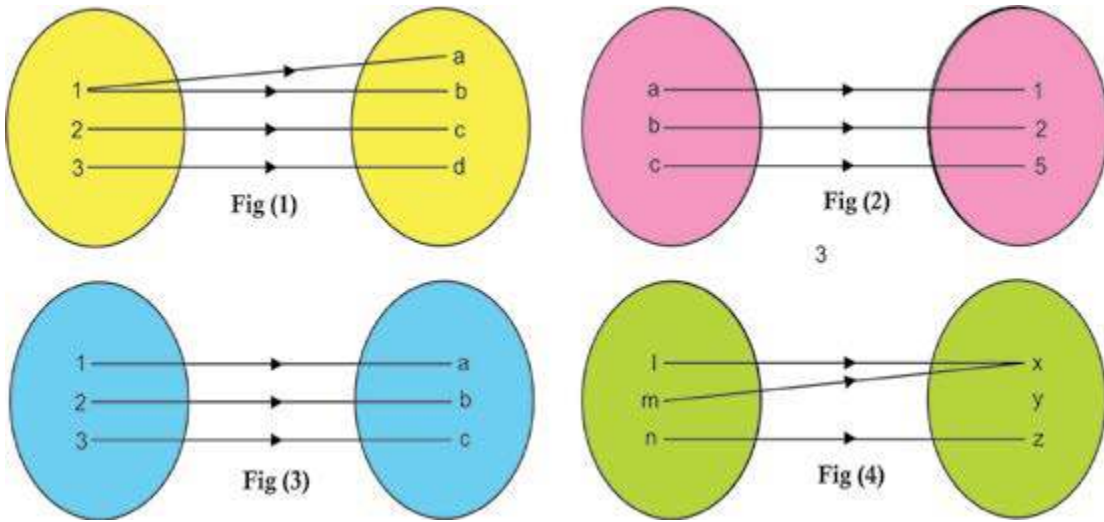
Exercise 2.6

1. For $A = \{1, 2, 3, 4\}$, find the following relations in A. State the domain and range of each relation. Also draw the graph of each.

i) $\{(x, y) \mid y = x\}$ ii) $\{(x, y) \mid y + x = 5\}$

ii) $\{(x, y) \mid x + y < 5\}$ iv) $\{(x, y) \mid x + y > 5\}$

2. Repeat *Q* -1 when $A = \mathbb{R}$, the set of real numbers. Which of the real lines are functions.

3. Which of the following diagrams represent functions and of which type?

4. Find the inverse of each of the following relations. Tell whether each relation and its inverse is a function or not: -

i) $\{(2,1), (3,2), (4,3), (5,4), (6,5)\}$ ii) $\{(1,3), (2,5), (3,7), (4,9), (5,11)\}$

iii) $\{(x, y) \mid y = 2x + 3, x \in \mathbb{R}\}$ iv) $\{(x, y) \mid y^2 = 4ax, x \geq 0\}$

v) $\{(x, y) \mid x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\}$

2.12 Binary Operations

In lower classes we have been studying different number systems investigating the properties of the operations performed on each system. Now we adopt the opposite course. We now study certain operations which may be useful in various particular cases.

An operation which when performed on a single number yields another number of the same or a different system is called a **unary operation.**

Examples of *Unary operations* are negation of a given number, extraction of square roots or cube roots of a number, squaring a number or raising it to a higher power.

We now consider binary operation, of much greater importance, operation which requires two numbers. We start by giving a formal definition of such an operation.

A *binary operation* denoted as ※ (read as star) on a non-empty set G is a function which associates with each ordered pair (a, b), of elements of G, a unique element, denoted as a ※ b of G.

In other words, a binary operation on a set G is a function from the set $G \times G$ to the set G. For convenience we often omit the word *binary* before *operation*.

Also in place of saying ※ is an operation on G, we shall say G is closed with respect to ※ .

Example 1: Ordinary addition, multiplication are operations on N. i.e., N is closed with respect to ordinary addition and multiplication because

$$\forall\, a, b \in N,\; a + b \in N \wedge a.b \in N$$

($\forall$ stands for" for all" and $\wedge$ stands for" and")

Example 2: Ordinary addition and multiplication are operations on E, the set of all even natural numbers. It is worth noting that addition is not an operation on O, the set of old natural numbers.

Example 3: With obvious modification of the meanings of the symbols, let E be any even natural number and O be any odd natural number, then

$E \oplus E = E$ (Sum of two even numbers is an even number).

$E \oplus O = O$

and $O \oplus O = E$

$\oplus$	E	O
E	E	O
O	O	E

These results can be beautifully shown in the form of a table given above:

This shows that the set $\{E, O\}$ is closed under (ordinary) addition.

The table may be read (horizontally).

$E \oplus E = E,\quad E \oplus O = O;$

$O \oplus O = E,\quad O \oplus E = O$

Example 4: The set $(1, -1, i, -i\}$ where $i = \sqrt{-1}$ is closed w.r.t multiplication (but not *w. r. t* addition). This can be verified from the adjoining table.

$\otimes$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Note: The elements of the set of this example are the **fourth roots** of unity.

Example 5: It can be easily verified that ordinary multiplication (but not addition) is an operation on the set $\{1, \omega, \omega^2\}$ where $\omega^3 = 1$. The adjoining table may be used for the verification of this fact.

(ω is pronounced omega)

$\otimes$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Operations on Residue Classes Modulo *n*.

Three consecutive natural numbers may be written in the form:

$3n$, $3n+1$, $3n+2$ When divided by 3 they give remainders 0, 1, 2 respectively.

Any other number, when divided by 3, will leave one of the above numbers as the reminder. On account of their special importance (in theory of numbers) the remainders like the above are called **residue classes Modulo 3.** Similarly, we can define **Residue classes Modulo 5** etc. An interesting fact about residue classes is that ordinary addition and multiplication are operations on such a class.

Example 6: Give the table for addition of elements of the set of residue classes modulo 5.

Solution: Clearly {0,1,2,3,4} is the set of residues that we have to consider. We add pairs of elements as in ordinary addition except that when the sum equals or exceeds 5, we divide it out by 5 and insert the remainder only in the table. Thus 4 + 3 = 7 but in place of 7 we insert 2(= 7−5) in the table and in place of 2 + 3 = 5 , we insert 0(= 5−5).

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Example 7: Give the table for addition of elements of the set of residue classes modulo 4.

Solution: Clearly {0,1,2,3}is the set of residues that we have to consider. We add pairs of elements as in ordinary addition except that when the sum equals or exceeds 4, we divide it out by 4 and insert the remainder only in the table. Thus 3 + 2 = 5 but in place

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

of 5 we insert 1(= 5−4) in the table and in place of 1 + 3 = 4 , we insert 0(= 4−4).

Example 8: Give the table for multiplication of elemnts of the set of residue classes modulo 4.

Solution: Clearly {0,1,2,3} is the set of residues that we have to consider. We multiply pairs of elements as in ordinary multiplcation except that when the product equals or exceeds 4, we divide it out by 4 and insert the remainder only in the table. Thus 3×2=6 but in place of 6 we insert 2 (= 6−4)in the table and in place of 2×2=4, we insert 0(= 4−4).

⊗	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Example 9: Give the table for multiplication of elements of the set of residue classes modulo 8.

Solution: Table is given below:

⊗	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Note: For performing multiplication of residue classes 0 is generally omitted.

2.12.1 Properties of Binary Operations

Let S be a non-empty set and ※ a binary operation on it. Then ※ may possess one or more of the following properties: -

i) Commutativity: ※ is said to be commutative if

$a ※ b = b ※ a \ \forall \ a,b \in S.$

ii) Associativity: ※ is said to be associative on S if

$a ※ (b ※ c) = (a ※ b) ※ c \ \forall \ a,b,c \in S.$

iii) Existence of an identity element: An element $e \in S$ is called an identity element w.r.t ※ if

$a ※ e = e ※ a = a, \ \forall \ a \in S.$

iv) Existence of inverse of each element: For any element $a \in S, \exists$ an element $a' \in S$ such that

$a ※ a' = a' ※ a = e$ (the identity element)

Note: **(1)** The Symbol ∃ stands for 'there exists'.

(2) Some authors include closure property in the properties of an operation. Since this property S is already included in the definition of *operation* we have considered it unnecessary to mention it in the above list.

(3) Some authors define *left identity* and *right identity* and also *left inverse* and *right inverse* of each element of a set and prove uniqueness of each of them. The following theorem gives their point of view: -

Theorem:

i) In a set S having a binary operation ※ a left identity and a right identity are the same.

ii) In a set having an associative binary Operation left inverse of an element is equal to its right inverse.

Proof:

ii) Let e' be the left identity and e'' be the right identity. Then

$e' ※ e'' = e'$ ($\because e''$ is a right identity)

$= e''$ ($\because e'$ is a left identity)

Hence $e' = e'' = e$

Therefore, e is the unique identity of S under ※

ii) For any $a \in S$, let a', a'' be its left and right inverses respectively then

$a' ※ (a ※ a'') = a' ※ e$ ($\because a''$ is right inverse of a)

$= a'$ ($\because e$ is the identity)

Also $(a' ※ a) ※ a'' = e ※ a''$ ($\because a'$ is left inverse of a)

$= a''$

But $a' ※ (a ※ a'') = (a' ※ a) ※ a''$ ※ is associative as supposed)

$\because \ a' = a''$

Inverse of a is generally written as a^{-1}.

Example 10: Let A =(1,2,3,...., 20}, the set of first 20 natural numbers.
Ordinary addition is not a binary operation on A because the set is not closed w.r.t. addition.
For instance, $10 + 25 = 25 \notin A$

Example 11: Addition and multiplication are commutative and associative operations on the sets

$N, Z, Q, \mathbb{R}$, (usual notation),

e.g. $4\times5 = 5\times4$, $2 + (3 + 5) = (2 + 3) + 5$ etc.

Example 12: Verify by a few examples that subtraction is not a binary operation on N but it is an operation on Z, the set of integers.

Exercise 2.7

1. Complete the table, indicating by a tick mark those properties which are satisfied by the specified set of numbers.

Property ↓ \ Set of numbers →		Natural	Whole	Integers	Rational	Reals
Closure	⊕					
	⊗					
Associative	⊕					
	⊗					
Identity	⊕					
	⊗					
Inverse	⊕					
	⊗					
Commutative	⊕					
	⊗					

2. What are the field axioms? In what respect does the field of real numbers differ from that of complex numbers?

3. Show that the adjoining table is that of multiplication of the elements of the set of residue classes modulo 5.

※	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

4. Prepare a table of addition of the elements of the set of residue classes modulo 4.

5. Which of the following binary operations shown in tables (a) and (b) is commutative?

※	*a*	*b*	*c*	*d*
a	*a*	*c*	*b*	*d*
b	*b*	*c*	*b*	*a*
c	*c*	*d*	*b*	*c*
d	*a*	*a*	*b*	*b*

(a)

※	*a*	*b*	*c*	*d*
a	*a*	*c*	*b*	*d*
b	*c*	*d*	*b*	*a*
c	*b*	*b*	*a*	*c*
d	*d*	*a*	*c*	*d*

(b)

6. Supply the missing elements of the third row of the given table so that the operation ※ may be associative.

※	*a*	*b*	*c*	*d*
a	*a*	*b*	*c*	*d*
b	*b*	*a*	*c*	*d*
c	-	-	-	-
d	*d*	*c*	*c*	*d*

7. What operation is represented by the adjoining table? Name the identity element of the relevant set, if it exists. Is the operation associative? Find the inverses of 0,1,2,3, if they exist.

※	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2.13 Groups

We have considered, at some length, binary operations and their properties. We now use our knowledge to classify sets according to the properties of operations defined on them.

First we state a few preliminary definitions which will culminate in the definition of a group.

Groupoid: A **groupoid** is a non-empty set on which a binary operation $※$ is defined.

Some authors call the system $(S, ※)$ a groupoid. But, for the sake of brevity and convenience we shall call S a **groupoid**, it being understood that an operation $※$ is defined on it.

In other words, a closed set with respect to an operation $※$ is called a groupoid.

Example 1: The set $\{E, O\}$ where E is any even number and O is any odd number, (as already seen) are closed w.r.t. addition.

It is, therefore, a groupoid.

Example 2: The set of Natural numbers is not closed under operation of subtraction e.g.,

For $4, 5 \in N$, $4 - 5 = -1 \notin N$

Thus $(N, -)$ is not a groupoid under subtraction.

Example 3: As seen earlier with the help of a table the set $\{1, -1, i, -i\}$, is closed w.r.t. . multiplication (but not w.r.t. addition). So it is also a groupoid w.r.t $\times$.

Semi-group: A non-empty set S is semi-group if;

i) It is **closed** with respect to an operation $※$ and

ii) The operation $※$ is **associative.**

As is obvious from its very name, a semi-group satisfies half of the conditions required for a group.

Example 4: The set of natural numbers, N, together with the operation of addition is a semi-group. N is clearly closed w.r.t. addition (+).

Also $\forall\ a,b,c \in N$, $\quad a + (b + c) = (a + b) + c$

Therefore, both the conditions for a semi-group are satisfied.

Non-commutative or non-abelian set: A set A is non-commutative if commutative law does not hold for it.

For example a set A is non-commutative or non-abilian set under $※$ when is defined as: $\forall\ x, y \in x ※ y = x$.

Clearly $x ※ y = x$ and $y ※ x = y$ indicates that A is non-commutative or non-abilian set.

Example 5: Consider Z, the set of integers together with the operation of multiplication. Product of any two integers is an integer.

Also product of integers is associative because $\forall a,b,c \in Z \quad a.(b.c) = (a.b).c$

Therefore, $(Z,.)$ is a semi-group.

Example 6: Let $P(S)$ be the power-set of S and let A, B, C, ... be the members of P. Since union of any two subsets of S is a subset of S, therefore P is closed with respect to $\cup$. Also the operation is associative.

(e.g. $A \cup (B \cup C) = (A \cup B) \cup C$, which is true in general),

Therefore, $\quad (P(S), \cup)$ is a semi-group.

Similarly $\quad (P(S), \cap)$ is a semi-group.

Example 7: Subtraction is non-commutative and non-associative on N.

Solution: For 4, 5, 6, $\in N$, we see that

$$4-5 = -1 \text{ and } 5-4 = 1$$

Clearly $\quad 4-5 \neq 5-4$

Thus subtraction is non-commutative on N.

Also $5-(4-1) = 5-(3) = 2$ and $(5-4)-1 = 1-1 = 0$

Clearly $5-(4-1) \neq (5-4)-1$

Thus subtraction is non-associative on N.

Example 8: For a set A of distinct elements, the binary operation $※$ on A defined by

$$x ※ y = x,\ \forall\ x, y \in A$$

is non commutative and assocaitve.

Solution : Consider

$x ※ y = x \quad$ and $y ※ x = y$

Clearly $\quad x ※ y \neq y ※ x$

Thus $※$ is non-commutative on A.

Monoid: A semi-group having an identity is called a **monoid** i.e., a monoid is a set S;

i) which is closed w.r.t. some operation $※$.

ii) the operation $※$ is associative and

iii) it has an identity.

Example 9: The power-set $P(S)$ of a set S is a **monoid** w.r.t. the operation $\cup$, because, as seen above, it is a semi-group and its identity is the empty-set Φ because if A is any subset of S,

$\Phi \cup A = A \cup \quad = A$

Example 10: The set of all non negative integers i.e., $Z^+ \cup \{0\}$

i) is clearly closed w.r.t. addition,

ii) addition is also associative, and

iii) 0 is the identity of the set.

$(a + 0 = 0 + a = a \quad \forall a \in Z^+ \cup \{0\})$

$\therefore$ the given set is a monoid w.r.t. addition.

Note: It is easy to verify that the given set is a monoid w.r.t. multiplication as well but not w.r.t. subtraction

Example 11: The set of natural numbers, N. w.r.t. $\otimes$

i) the product of any two natural numbers is a natural number;

ii) Product of natural numbers is also associative i.e.,

$\forall a, b, c \in N \quad a.(b.c) = (a.b).c$

iii) $1 \in N$ is the identity of the set.

$\therefore N$ is a monoid w.r.t. multiplication

Note: N is not a monoid w.r.t. addition because it has no identity w.r.t. addition.

Definition of Group: A *monoid* having inverse of each of its elements under ※ is called a group under ※. That is a group under ※ is a set G (say) if

i) G is closed w.r.t. some operation ※

ii) The operation of ※ is associative;

iii) G has an identity element w.r.t. ※ and

iv) Every element of G has an inverse in G w.r.t. ※.

If G satisfies the additional condition:

v) For every $a, b \in G$

a ※ $b = b$ ※ a

then G is said to be an Abelian* or commutative group under ※

Example 12: The set N w.r.t. +

Condition (i) colsure: satisfied i.e., $\forall\ a, b \in N, a + b \in N$

(ii) Associativity: satisfied i.e.,

$\forall\ a,b,c \in N, a + (b + c) = (a + b) + c$

(iii) and (iv) not satisfied i.e., neither identity nor inverse of any element exists.

$\therefore N$ is only a semi-group. Neither *monoid* nor a *group w.r.t.* +.

Example 13: N w.r.t $\otimes$

Condition: (i) Closure: satisfied

$\forall a, b \in N, \quad a, b \in N$

(ii) Associativity: satisfied

$\forall\ a,b,c \in N, \quad a.(b.c) = (a.b).c$

(iii) Identity element, yes, 1 is the identity element

(iv) Inverse of any element of N does not exist in N, so N is a *monoid* but not a group under multiplication.

Example 14: Consider $S = \{0,1,2\}$ upon which operation $\oplus$ has been performed as shown in the following table. Show that S is an abelian group under $\oplus$.

Solution :

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

i) Clearly S as shown under the operation is closed.

ii) The operation is associative e.g

$0 + (1 + 2) = 0 + 0 = 0$

$(0 + 1) + 2 = 1 + 2 = 0$ etc.

iii) Identity element 0 exists.

iv) Inverses of all elements exist, for example

$0 + 0 = 0,\ 1 + 2 = 0,\ 2 + 1 = 0$

$\Rightarrow 0^{-1} = 0 \quad 1^{-1} = 2,\ 2^{-1} = 1$

v) Also $\oplus$ is clearly commutative e.g., $1 + 2 = 0 = 2 + 1$

Hence the result,

Example 15: Consider the set $S = \{1, -1, i\ -i\}$. Set up its multiplication table and show that the set is an abelian group under multiplication

Solution :

⊗	1	−1	i	$-i$
1	1	-1	i	$-i$
−1	−1	1	$-i$	i
i	i	$-i$	−1	1
$-i$	$-i$	i	1	−1

i) S is evidently closed w.r.t.⊗.

ii) Multiplication is also associative

(Recall that multiplication of complex numbers is associative)

iii) Identity element of S is 1.

iv) Inverse of each element exists.

Each of 1 and -1 is self inverse.

i and - i are inverse of each other.

v) ⊗ is also commutative as in the case of C, the set of complex numbers. Hence given set is an *Abelian group*.

Example : Let G be the set of all 2×2 non-singular real matrices, then under the usual multiplication of matrices, G is a non-abelian group.

Condition (i) Closure: satisfied; i.e., product of any two 2×2 matrices is again a matrix of order 2×2.

(ii) Associativity: satisfied

For any matrices A , B and C conformable for multiplication.

$$A \times (B \times C) = (A \times B) \times C.$$

So, condition of associativity is satisfied for 2×2 matrices

(iii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix.

(iv) As G contains non-singular matrices only so, it contains inverse of each of its elements.

(v) We know that $AB \neq BA$ in general. Particularly for G, $AB \neq BA$.

Thus G is a non-abilian or non-commutative gorup.

Finite and Infinite Gorup: A gorup G is said to be a **finite group** if it contains finite number of elements. Otherwise G is an **infinite group**.

The given examples of groups are clearly distinguishable whether finite or infinite.

Cancellation laws: If a,b,c are elements of a group G, then

i) $ab = ac \Rightarrow b = c$ (Left cancellation Law)

ii) $ba = ca \Rightarrow b = c$ (Right cancellation Law)

Proof: (i) $ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$ (by associative law)

$\Rightarrow eb = ec$ $(\because a^{-1}a = e)$

$\Rightarrow b = c$

ii) Prove yourselves.

2.14 Solution of linear equations

a,b being elements of a group G, solve the following equations:

i) $ax = b$, ii) $xa = b$

Solution : (i) Given: $ax = b \Rightarrow a^{-1}(ax) = a^{-1}b$

$\Rightarrow (a^{-1}a)\,x = a^{-1}b$ (by associativity)

$\Rightarrow ex = a^{-1}b$

$\Rightarrow x = a^{-1}b$ which is the desired solution.

ii) Solve yourselves.

Note: Since the inverse (left or right) of any element a of a group is unique, from the above procedure, it follows that the above solution is also unique.

2.15 Reversal law of inverses

If a,b are elements of a group G, then show that

$$(ab)^{-1} = b^{-1}a^{-1}$$

Proof: $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$ (Associative law)

$= a\,e\,a^{-1}$

$= aa^{-1}$

$= e$

$\therefore ab$ and $b^{-1}a^{-1}$ are inverse of each other.

Note: The rule can obviously be extended to the product of three or more elements of a group.

Theorem: If (G, ※) is a group with *e* its identity, then *e* is unique.

Proof: Suppose the contrary that identity is not unique. And let *e*′ be another identity.

e, *e*′ being identities, we have

$e' ※ e = e ※ e' = e'$ (*e* is an identity) (i)

$e' ※ e = e ※ e' = e$ (*e*′ is an identity) (ii)

Comparing (i) and (ii)

$e' = e$.

Thus the identity of a group is always unique.

Examples:

i) (*Z*, +) has no identity other then 0 (zero).

ii) ($\mathbb{R}$ – {0}, ×) has no identity other than 1.

iii) (*C*,+) has no identity other than 0 + 0*i*.

iv) (*C*,.) has no identity other than 1 + 0*i*.

v) (M_2,.) has no identity other than $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

where M_2 is a set of 2 × 2 matrices.

Theoram: If (*G*, ※) is a group and $a \in G$, there is a unique inverse of *a* in *G*.

Proof: Let (*G*, ※) be a group and $a \in G$.

Suppose that *a*′ and *a*″ are two inverses of *a* in *G*. Then

$a' = a' ※ e = a' ※ (a ※ a'')$ (*a*″is an inverse of *a* w.r.t. ※)

$= (a' ※ a) ※ a''$ (Associative law in *G*).

$= e ※ a''$ (*a*′ is an inverse of *a*).

$= a''$ (*e* is an identity of *G*).

Thus inverse of *a* is unique in *G*.

Examples 16:

i) in group (*Z*, +), inverse of 1 is -1 and inverse of 2 is –2 and so on.

ii) in group ($\mathbb{R}$ – {0}, ×) inverse of 3 is $\frac{1}{3}$ etc.

Exercise 2.8

1. Operation ⊕ performed on the two-member set *G* = {0,1}is shown in the adjoining table. Answer the questions: -

⊕	0	1
0	0	1
1	1	0

i) Name the identity element if it exists?

ii) What is the inverse of 1 ?

iii) Is the set *G*, under the given operation a group? Abelian or non-Abelian?

2. The operation ⊕ as performed on the set {0,1,2,3} is shown in the adjoining table, show that the set is an Abelian group?

⊕	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

3. For each of the following sets, determine whether or not the set forms a group with respect to the indicated operation.

	Set	Operation
i)	The set of rational numbers	×
ii)	The set of rational numbers	+
iii)	The set of positive rational numbers	×
iv)	The set of integers	+
v)	The set of integers	×

4. Show that the adjoining table represents the sums of the elements of the set {*E, O*}. What is the identity element of this set? Show that this set is an *abelian group*.

⊕	*E*	*O*
E	*E*	*O*
O	*O*	*E*

5. Show that the set $\{1, \omega, \omega^2\}$, when $\omega^3 = 1$, is an Abelian group w.r.t. ordinary multiplication.

6. If *G* is a group under the operation and $a, b \in G$, find the solutions of the equations: $a ※ x = b$, $x ※ a = b$

7. Show that the set consisting of elements of the form $a + \sqrt{3}\, b$ (*a, b* being rational), is an abelian group w.r.t. addition.

8. Determine whether,(*P*(*S*), ※), where ※ stands for intersection is a semi-group, a monoid

or neither. If it is a monoid, specify its identity.

9. Complete the following table to obtain a semi-group under ※

※	a	b	c
a	c	a	b
b	a	b	c
c	-	-	a

10. Prove that all 2×2 non-singular matrices over the real field form a non-abelian group under multiplication.

CHAPTER

3

Matrices and Determinants

Animation 3.1: Addition of matrix
Source & Credit: elearn.punjab

3.1 Introduction

While solving linear systems of equations, a new notation was introduced to reduce the amount of writing. For this new notation the word matrix was first used by the English mathematician James Sylvester (1814 - 1897). Arthur Cayley (1821 - 1895) developed the theory of matrices and used them in the linear transformations. Now-a-days, matrices are used in high speed computers and also in
other various disciplines.

The concept of determinants was used by Chinese and Japanese but the Japanese mathematician Seki Kowa (1642 - 1708) and the German Mathematician Gottfried Wilhelm Leibniz (1646 - 1716) are credited for the invention of determinants. G. Cramer (1704 - 1752) applied the determinants successfully for solving the systems of linear equations.

A rectangular array o f numbers enclosed by a pair o f brackets such as:

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix} \quad \text{(i)} \quad \text{or} \quad \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 2 & 6 \\ 4 & 1 & -1 \end{bmatrix} \quad \text{(ii)}$$

is called a **matrix**. The horizontal lines of numbers are called **rows** and the vertical lines of numbers are called **columns**. The numbers used in rows or columns are said to be the **entries** or **elements** of the matrix.

The matrix in (i) has two rows and three columns while the matrix in (ii) has 4 rows and three columns. Note that the number of elements of the matrix in (ii) is $4 \times 3 = 12$. Now we give a general definition of a matrix.

Generally, a bracketed rectangular array of $m \times n$ elements
a_{ij} ($i = 1, 2, 3, ..., m$; $j = 1, 2, 3, ..., n$), arranged in m rows and n columns such as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad \text{(iii)}$$

is called an m by n matrix (written as $m \times n$ matrix).

$m \times n$ is called the order of the matrix in (iii). We usually use capital letters such as A, B, C, X, Y, etc., to represent the matrices and small letters such as $a, b, c, ... l, m, n, ..., a_{11}, a_{12}, a_{13}$,...., etc., to indicate the entries of the matrices.

Let the matrix in (iii) be denoted by A. The ith row and the jth column of A are indicated in the following tabular representation of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \quad \text{(iv)}$$

(jth column ↓ marks the jth column; ith row → marks the ith row)

The elements of the ith row o f A are a_{i1} a_{i2} a_{i3} ...... a_{ij} ...... a_{in}
while the elements of the jth column of A are a_{1j} a_{2j} a_{3j} .... a_{ij} ...... a_{mj} .
We note that a_{ij} is the element of the ith row and jth column of A. The double subscripts are useful to name the elements of the matrices. For example, the element 7 is at a_{23} position in

the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$

$A = [a_{ij}]_{m \times n}$ or $A = [a_{ij}]$, for $i = 1, 2, 3,...., m$; $j = 1, 2, 3,...., n$, where a_{ij} is the element of the ith row and jth column of A.

Note: a_{ij} is also known as the (i, j)th element or entry of A.

The elements (entries) of matrices need not always be numbers but in the study of matrices, we shall take the elements o f the matrices from $\Re$ or from C.

Note: The matrix A is called real if all of its elements are real.

Row Matrix or Row vector: A matrix, which has only one row, i.e., a $1 \times n$ matrix of the form $[a_{i1} \ \ a_{i2} \ \ a_{i3} \ \ldots\ldots \ a_{in}]$ is said to be a row matrix or a row vector.

Column Matrix or Column Vector: A matrix which has only one column i.e., an $m \times 1$ matrix of the form $\begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is said to be a column matrix or a column vector.

For example $[1 \ \ -1 \ \ 3 \ \ 4]$ is a row matrix having 4 columns and $\begin{bmatrix} \\ \\ \end{bmatrix}$ is a column matrix having 3 rows.

Rectangular Matrix : If m ≠ n, then the matrix is called a rectangular matrix of order $m \times n$, that is, the matrix in which the number of rows is not equal to the number of columns, is said to be a rectangular matrix.

For example; $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & 4 \\ 3 & -1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ are rectangular matrices of orders 2×3 and 4×3 respectively.

Square Matrix : If $m = n$, then the matrix of order $m \times n$ is said to be a square matrix of order n or m. i.e., the matrix which has the same number of rows and columns is called a square matrix. For example;

$[\,0\,]$, $\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 8 \\ 3 & 5 & 4 \end{bmatrix}$ are square matrices of orders 1, 2 and 3 respectively.

Let $A = [a_{ij}]$ be a square matrix o f order n, then the entries a_{11}, a_{22}, a_{33}, ..., a_{nn} form the principal diagonal for the matrix A and the entries a_{1n}, $a_{2\,n-1}$, $a_{3\,n-2}$, ..., $a_{n-1\,2}$, a_{n1} form the secondary diagonal for the

matrix A. For example, $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, in the matrix the entries of the principal diagonal are a_{11}, a_{22}, a_{33}, a_{44} and the entries of the secondary diagonal are a_{14}, a_{23}, a_{32}, a_{41}.

The principal diagonal of a square matrix is also called the leading diagonal or main diagonal of the matrix.

Diagonal Matrix: Let $A = [a_{ij}]$ be a square matrix of order n.

If $a_{ij} = 0$ for all $i \neq j$ and at least one $a_{ij} \neq 0$ for $i = j$, that is, some elements of the principal diagonal of A may be zero but not all, then the matrix A is called a diagonal matrix. The matrices

$[7]$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ are diagonal matrices.

Scalar Matrix: Let $A = [a_{ij}]$ be a square matrix of order n.

If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = k$ (some non-zero scalar) for all $i = j$, then the matrix A is called a scalar matrix of order n. For example;

$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$, $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ are scalar matrices of order 2, 3 and 4 respectively.

Unit Matrix or Identity Matrix : Let $A = [a_{ij}]$ be a square matrix of order n. If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$, then the matrix A is called a *unit matrix* or *identity matrix* of order n. We denote such matrix by I_n and it is of the form:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

The identity matrix of order 3 is denoted by I_3, that is, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Null Matrix or Zero Matrix : A square or rectangular matrix whose each element is zero, is called a null or zero matrix. An m × n matrix with all its elements equal to zero, is denoted by $O_{m \times n}$. Null matrices may be of any order. Here are some examples:

$$[0], [0 \quad 0 \quad 0], \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

O may be used to denote null matrix of any order if there is no confusion.

Equal Matrices: Two matrices of the same order are said to be equal if their corresponding entries are equal. For example, $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal, i.e., $A = B$ iff $a_{ij} = b_{ij}$ for $i = 1, 2, 3, \ldots, m$, $j = 1, 2, 3, \ldots, n$. In other words, *A* and *B* represent the same matrix.

3.1.1 Addition of Matrices

Two matrices are conformable for addition if they are of the same order. The sum $A + B$ of two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the $m \times n$ matrix $C = [c_{ij}]$ formed by adding the corresponding entries of *A* and *B* together. In symbols, we write as $C = A + B$, that is: $[c_{ij}] = [a_{ij} + b_{ij}]$

where $c_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

Note that $a_{ij} + b_{ij}$ is the (i, j)th element of $A + B$.

Transpose of a Matrix:

If *A* is a matrix of order $m \times n$ then an $n \times m$ matrix obtained by interchanging the rows and columns of *A*, is called the transpose of *A*. It is denoted by A^t. If $[a_{ij}]_{m \times n}$ then the transpose of *A* is defined as:

$A^t = [a'_{ij}]_{n \times m}$ where $a'_{ij} = a_{ji}$ .. for $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, m$

For example, if $B = [b_{ij}]_{3 \times 4} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$, then

$B^t = [b'_{ij}]_{4 \times 3}$ where $b'_{ij} = b_{ji}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ i.e.,

$$B^t = \begin{bmatrix} b'_{11} & b'_{12} & b'_{13} \\ b'_{21} & b'_{22} & b'_{23} \\ b'_{31} & b'_{32} & b'_{33} \\ b'_{41} & b'_{42} & b'_{43} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \\ b_{14} & b_{24} & b_{34} \end{bmatrix}$$

Note that the 2nd row of *B* has the same entries respectively as the 2nd column of B^t and the 3rd row of B^t has the same entries respectively as the 3rd column of *B* etc.

Example 1:

If $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$, then show that

$(A + B)^t = A^t + B^t$

Solution :

$$A + B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+2 & 0+(-1) & -1+3 & 2+1 \\ 3+1 & 1+3 & 2+(-1) & 5+4 \\ 0+3 & -2+1 & 1+2 & 6+(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 & 3 \\ 4 & 4 & 1 & 9 \\ 3 & -1 & 3 & 5 \end{bmatrix}$$

and $(A+B)^t = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix}$ (1)

Taking transpose of A and B, we have

$$A^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} \text{ and } B^t \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix}, \text{ so}$$

$$A^t + B^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 2 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 3 \\ -1 & 4 & -1 \\ 2 & 1 & 3 \\ 3 & 9 & 5 \end{bmatrix} \quad (2)$$

From (1) and (2), we have $(A+B)^t = A^t + B^t$

3.1.2 Scalar Multiplication

If $A = [a_{ij}]$ is $m \times n$ matrix and k is a scalar, then the product of k and A, denoted by kA, is the matrix formed by multiplying each entry of A by k, that is,

$$kA = [ka_{ij}]$$

Obviously, order of kA is $m \times n$.

Note. If n is a positive integer, then $A + A + A + \ldots$ to n times $= nA$.

If $A = [a_{ij}] \in M_{m\times n}$ (the set of all $m \times n$ matrices over the real field $\Re$ then $ka_{ij} \in \Re$, for all i and j, which shows that $kA \in M_{m\times n}$. It follows that the set $M_{m\times n}$ possesses the closure property with respect to scalar multiplication. If $A, B \in M$ and r,s are scalars, then we can prove that $r(sA) = (rs)A$, $(r+s)A = rA + sA$, $r(A+B) = rA + rB$.

3.1.3 Subtraction of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, then we define subtraction of B from A as:

$$A - B = A + (-B)$$
$$= [a_{ij}] + [-b_{ij}] = [a_{ij} - b_{ij}] \text{ for } i = 1, 2, 3, \ldots, m;\ j = 1, 2, 3, \ldots, n$$

Thus the matrix $A - B$ is formed by subtracting each entry of B from the corresponding entry of A.

3.1.4 Multiplication of two Matrices

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]$ be a 2×3 matrix and $B = [b_{ij}]$ be a 3×2 matrix. Then the product AB is defined to be the 2×2 matrix C whose element c_{ij} is the sum of products of the corresponding elements of the ith row of A with elements of jth column of B. The element c_{21} of ***C*** is shown in the figure (A), that is

First column of B

2nd row of A

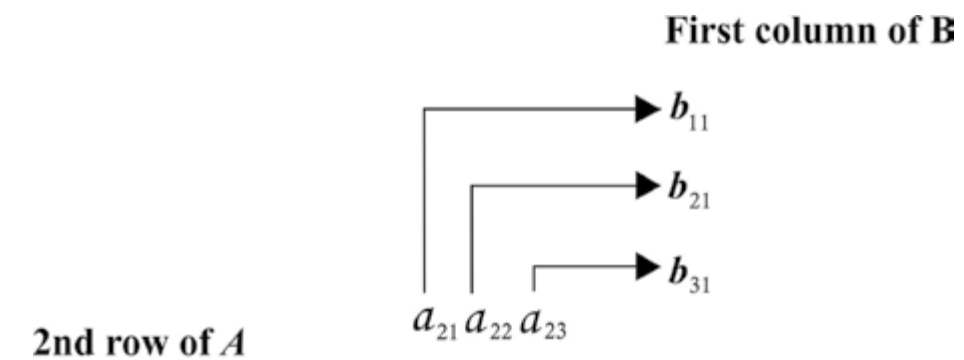



Fig.(A)

$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$. Thus

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

Similarly $BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$$= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \\ b_{31}a_{11} + b_{32}a_{21} & b_{31}a_{12} + b_{32}a_{22} & b_{31}a_{13} + b_{32}a_{23} \end{bmatrix}$$

AB and BA are defined and their orders are 2×2 and 3×3 respectively.

Note 1. Both products AB and BA are defined but $AB \neq BA$

2. If the product AB is defined, then the order of the product can be illustrated as given below:

Order of A $\quad m \times n$

Order of B $\quad n \times p$

Order of AB $\quad m \times p$

Example 2: If $A=\begin{bmatrix}2 & -1 & 0\\ 1 & 2 & -3\\ 1 & 2 & -2\end{bmatrix}$ and $B=\begin{bmatrix}2 & -2 & 3\\ -1 & -4 & 6\\ 0 & -5 & 5\end{bmatrix}$, then compute A^2B.

Solution :

$$A^2 = AA \begin{bmatrix}2 & -1 & 0\\ 1 & 2 & 3\\ 1 & 2 & -2\end{bmatrix}\begin{bmatrix}2 & -1 & 0\\ 1 & 2 & 3\\ 1 & 2 & -2\end{bmatrix}$$

$$=\begin{bmatrix}4-1+0 & -2-2+0 & 0+3+0\\ 2+2-3 & -1+4-6 & 0-6+6\\ 2+2-2 & -1+4-4 & 0-6+4\end{bmatrix}=\begin{bmatrix}3 & -4 & 3\\ 1 & -3 & 0\\ 2 & -1 & -2\end{bmatrix}$$

$$\therefore A^2B=\begin{bmatrix}3 & -4 & 3\\ 1 & -3 & 0\\ 2 & -1 & -2\end{bmatrix}\begin{bmatrix}2 & -2 & 3\\ -1 & -4 & 6\\ 0 & -5 & 5\end{bmatrix}$$

$$=\begin{bmatrix}6+4+0 & -6+16-15 & 9-24+15\\ 2+3+0 & -2+12+0 & 3-18+0\\ 4+1+0 & -4+4+10 & 6-6-10\end{bmatrix}=\begin{bmatrix}10 & -5 & 0\\ 5 & 10 & -15\\ 5 & 10 & -10\end{bmatrix}$$

Note: Powers of square matrices are defined as:

$A^2 = A \times A$, $A^3 = A \times A \times A$,

$A^n = A \times A \times A \times$ .... to n factors.

3.2 Determinant of a 2 × 2 matrix

We can associate with every square matrix A over $\Re$ or C, a number $|A|$, known as the determinant of the matrix A.

The determinant of a matrix is denoted by enclosing its square array between vertical bars instead of brackets. The number of elements in any row or column is called the order of determinant. For example,

if $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then the determinant of A is $\begin{vmatrix}a & b\\ c & d\end{vmatrix}$. Its value is defined to be the real number $ad - bc$, that is,

$$|A|=\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$$

For example, if $A=\begin{bmatrix}2 & -1\\ 4 & 3\end{bmatrix}$ and $B\ \begin{bmatrix}1 & 4\\ 2 & 8\end{bmatrix}$, then

$$|A|=\begin{vmatrix}2 & -1\\ 4 & 3\end{vmatrix} = (2)(3) - (-1)(4) = 6 + 4 = 10$$

and $$|B|=\begin{vmatrix}1 & 4\\ 2 & 8\end{vmatrix} = (1)(8) - (4)(2) = 8 - 8 = 0$$

Hence the determinant of a matrix is the difference of the products of the entries in the two diagonals.

$$\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc$$

$-bc \quad ad$

Note: The determinant of a 1×1 matrix $[a_{11}]$ is defined as $|a_{11}| = a_{11}$

3.2.1 Singular and Non-Singular Matrices

A square matrix A is **singular** if $|A| = 0$, otherwise it is a **non singular** matrix. In the above example, $|B| = 0 \Rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ is a singular matrix and $|A| = 10 \neq 0 \Rightarrow A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ is a non-singular matrix.

3.2.2 Adjoint of a 2 × 2 Matrix

The adjoint of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by *adj A* and is defined as: *adj* $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3.2.3 Inverse of a 2 × 2 Matrix

Let A be a non-singualr square matrix of **order 2**. If there exists a matrix B such that $AB = BA = I_2$ where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then B is called the multiplicative inverse of A and is usually denoted by A^{-1}, that is, $B = A^{-1}$

$$\boxed{\text{Thus } AA^{-1} = A^{-1}A = I_2}$$

Example 3: For a non-singular matrix A, prove that $A = \frac{1}{|A|} adjA$

Solution : If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, Then:

$AA^{-1} = I_2$, that is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} ap+br & ap+bs \\ cp+dr & cq+ds \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} ap+br=1....(i) = \quad ap \quad bs \quad 0....(ii) \\ cp+dr=0....(iii) = \quad cq \quad ds \quad 1....(iv) \end{cases}$$

From (iii), $r = \frac{-c}{d}p$

Putting the value of r in (i), we have

$$ap + b\left(\frac{-c}{d}p\right) = 1 \Rightarrow \left(\frac{ad-bc}{d}\right)p = 1 \Rightarrow p = \frac{d}{ad-bc}$$

and $\quad r = \frac{-c}{d}p = \frac{-c}{d}.\frac{d}{ad-bc} = -\frac{c}{ad-bc}$

Similarly, solving (ii) and (iv), we get

$$q = \frac{-b}{ad-bc}, \quad s = \frac{a}{ad-bc}$$

Substituting these values in $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, we have

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus $A^{-1} = \frac{1}{|A|} AdjA$

Example 4: Find A^{-1} if $A=\begin{bmatrix}5 & 3\\1 & 1\end{bmatrix}$ and verify that $AA^{-1}=A^{-1}A$

Solution : $|A|=\begin{vmatrix}5 & 3\\1 & 1\end{vmatrix}=5-3=2$

Since $|A|\neq 0$, we can find A^{-1}.

$$A^{-1}=\frac{1}{|A|}AdjA$$

$$\Rightarrow\quad A^{-1}=\frac{1}{2}\begin{bmatrix}1 & -3\\-1 & 5\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{-3}{2}\\\frac{-1}{2} & \frac{5}{2}\end{bmatrix}$$

Now $\quad A.A^{-1}=\begin{bmatrix}5 & 3\\1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{-3}{2}\\\frac{-1}{2} & \frac{5}{2}\end{bmatrix}$

$$=\begin{bmatrix}\frac{5}{2}-\frac{3}{2} & \frac{-15}{2}+\frac{15}{2}\\\frac{1}{2}-\frac{1}{2} & \frac{-3}{2}+\frac{5}{2}\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\qquad\text{(i)}$$

and $\quad A^{-1}.A=\begin{bmatrix}\frac{1}{2} & \frac{-3}{2}\\\frac{-1}{2} & \frac{5}{2}\end{bmatrix}\begin{bmatrix}5 & 3\\1 & 1\end{bmatrix}$

$$=\begin{bmatrix}\frac{5}{2}-\frac{3}{2} & \frac{3}{2}-\frac{3}{2}\\\frac{-5}{2}+\frac{5}{2} & \frac{-3}{2}+\frac{5}{2}\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\qquad\text{(ii)}$$

From (i) and (ii), we have

$$AA^{-1}=A^{-1}A$$

3.3 Solution of simultaneous linear equations by using matrices

Let the system of linear equations be

$$\left.\begin{aligned}a_{11}x_1+a_{12}x_2=b_1\\a_{21}x_1+a_{22}x_2=b_2\end{aligned}\right\}\qquad\text{(i)}$$

where a_{11}, a_{12}, a_{21}, a_{22}, b_1 and b_1 are real numbers.
The system (i) can be written in the matrix form as:

$$\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix}\quad\text{or}\quad AX=B\qquad\text{(ii)}$$

$$\text{where}\quad A=\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix},X=\begin{bmatrix}x_1\\x_2\end{bmatrix}B=\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

If $|A|\neq 0$, then A^{-1} exists so (ii) gives

$A^{-1}(AX)=A^{-1}B$ (By pre-multiplying (ii) by A^{-1})

or $(A^{-1}A)X=A^{-1}B$ (Matrix multiplication is associative)

$\Rightarrow\ X=A^{-1}B$ ($\because A^{-1}A=I_2$)

$\Rightarrow\ X=A^{-1}B$

$$\text{or}\quad\begin{bmatrix}x_1\\x_2\end{bmatrix}=\frac{1}{|A|}\begin{bmatrix}a_{22} & -a_{12}\\-a_{21} & a_{11}\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$$=\frac{1}{|A|}\begin{bmatrix}a_{22}b_1-a_{12}b_2\\-a_{21}b_1-a_{11}b_2\end{bmatrix}\begin{bmatrix}\frac{b_1a_{22}-a_{12}b_2}{|A|}\\\frac{a_{11}b_2-b_1a_{21}}{|A|}\end{bmatrix}$$

$$\text{Thus}\quad=\frac{\begin{vmatrix}b_1 & a_{12}\\b_2 & a_{22}\end{vmatrix}}{|A|}\text{ and }x_2\ \frac{\begin{vmatrix}a_{11} & b_1\\b_{21} & b_2\end{vmatrix}}{|A|}$$

It follows from the above discussion that the system of linear equations such as (i) has a unique solution if $|A|\neq 0$.

Example 5: Solve the following systems of linear equations.

i) $\left.\begin{matrix} 3x_1 - x_2 = 1 \\ x_1 + x_2 = 3 \end{matrix}\right\}\begin{matrix} + \quad = \\ + \quad = \end{matrix}$ ii) $\left.\begin{matrix} x_1 & 2x_2 & 4 \\ 2x_1 & 4x_2 & 12 \end{matrix}\right\}$

Solution : (i) The matrix form of the system $\left.\begin{matrix} 3x_1 - x_2 = 1 \\ x_1 + x_2 = 3 \end{matrix}\right\}$ is

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

or $AX = B$ . . . . (i) where $A == \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, $X \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $B \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = 3 + 1 = 4$$

and adj $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, therefore,

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

(I) becomes $X = A^{-1}B$, that is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & +\frac{3}{4} \\ -\frac{1}{4} & +\frac{9}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\Rightarrow$ $x_1 = 1$ and $x_2 = 2$

(ii) The matrix form of the system $\left.\begin{matrix} x_1 + 2x_2 = 4 \\ 2x_1 + 4x_2 = 12 \end{matrix}\right\}$ is

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$$

and $|A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$ = 4 – 4 = 0, so A^{-1} does not exist.

Multiplying the first equation o f the above system by 2, we have

$2x_1 + 4x_2 = 8$ but $2x_1 + 4x_2 = 12$

which is impossible. Thus the system has no solution.

Exercise 3.1

1. If $A == \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and $B \begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix}$, then show that

i) $4A - 3A = A$ ii) $3B - 3A = 3(B - A)$

2. If $A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$, show that $A^4 = I_2$.

3. Find x and y if

i) $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} == \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

4. If $A == \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ and $B \begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$, find the following matrices;

i) $4A - 3B = A$ ii) $A + 3(B - A)$

5. Find *x* and y If $\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & x & y \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

6. If $A = [a_{ij}]_{3\times3}$, find the following matrices;

i) $\lambda(\mu A) = (\lambda\mu)A$ ii) $(\lambda+\mu)A = \lambda A + \mu A$ ii) $\lambda A - A = (\lambda-1)A$

7. If $A = [a_{ij}]_{2\times3}$ and $B = [b_{ij}]_{2\times3}$, show that $\lambda(A+B) = \lambda A + \lambda B$.

8. If $A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$ and $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, find the values of a and b.

9. If $A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$ and $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find the values of a and b.

10. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$, then show that $(A + B)^t = A^t + B^t$.

11. Find A^3 if $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

12. Find the matrix X if;.

i) $X\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}X = \begin{bmatrix} 2 & 1 \\ 5 & 10 \end{bmatrix}$

13. Find the matrix A if,

i) $\begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 3 & 1 \end{bmatrix}A = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}A = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$

14. Show that $\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi \end{bmatrix} = rI_3$.

3.4 Field

A set F is called a **field** if the operations of addition '+' and multiplication '.' on F satisfy the following properties written in tabular form as:

	Addition	Multiplication
Closure		
i)	For any $a,b \in F$, $a + b \in F$	For any $a,b \in F$, $a.b \in F$
Commutativity		
ii)	For any $a,b \in F$, $a + b = b + a$	For any $a,b \in F$, $a.b = b.a$
Associativity		
iii)	For any $a,b,c \in F$, $(a + b) + c = a + (b + c)$	For any $a,b,c \in F$, $(a.b).c = a.(b.c)$
Existence of Identity		
iv)	For any $a \in F, \exists\ 0 \in F$ such that $a + 0 = 0 + a = a$	For any $a \in F, \exists\ 1 \in F$ such that $a.1 = 1.a = a$
Existence of Inverses		
v)	For any $a \in F, \exists -a \in F$ such that $a + (-a) = (-a) + a = 0$	v) For any $a \in F$, $a \neq 0$ $\exists \frac{1}{a} \in F$ such that $a.(\frac{1}{a}) = (\frac{1}{a}).a = 1$
Distributivity		
vi)	For any $a,b,c \in F$,	D_1 : $a(b + c) = ab + ac$ D_2 : $(b + c)a = ba + ca$

All the above mentioned properties hold for Q, $\Re$, and C.

3.5 Properties of Matrix Addition , Scalar Multiplication and Matrix Multiplication.

If A, B and C are $n\times n$ matrices and c and d are scalers, the following properties are true:

1. **Commutative property w.r.t. addition:** $A + B = B + A$

Note: w.r.t. is used for "with respect to".

2. **Associative property w.r.t. addition:** $(A + B) + C – A + (B + C)$
3. **Associative property of scalar multiplication:** $(cd)A = c(dA)$
4. **Existance of additive identity:** $A + O = O + A – A$ (O is null matrix)
5. **Existance of multiplicative identity:** $IA = AI = A$ (I is unit/identity matrix)
6. **Distributive property w.r.t scalar multiplication:**
   (a) $c(A + B) = cA + cB$ **(b)** $(c + d)A = cA + dA$
7. **Associative property w.r.t. multiplication:** $A(BC) = (AB)C$
8. **Left distributive property:** $A(B + C) = AB + AC$
9. **Right distributive property:** $(A + B)C = AC + BC$
10. $c(AB) = (cA)B = A(cB)$

Example 1: Find AB and BA if $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 1 & -2 & 3 \end{bmatrix}$

Solution : $AB = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 1 & -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2\times1+0\times2+1\times1 & 2\times(-1)+0\times3+1\times(-2) & 2\times0+0\times(-1)+1\times3 \\ 1\times1+4\times2+2\times1 & 1\times(-1)+4\times3+2\times(-2) & 1\times0+4\times(-1)+2\times3 \\ 3\times1+0\times2+6\times1 & 3\times(-1)+0\times3+6\times(-2) & 3\times0+0\times(-1)+6\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 3 \\ 11 & 7 & 2 \\ 9 & -15 & 18 \end{bmatrix} \qquad (1)$$

$$BA = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times2+(-1)\times1+0\times3 & 1\times0+(-1)\times4+0\times0 & 1\times1+(-1)\times2+0\times6 \\ 2\times2+3\times1+(-1)\times3 & 2\times0+3\times4+(-1)\times0 & 2\times1+3\times2+(-1)\times6 \\ 1\times2+(-2)\times1+3\times3 & 1\times0+(-2)\times4+3\times0 & 1\times1+(-2)\times2+3\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & -1 \\ 4 & 12 & 2 \\ 9 & -8 & 15 \end{bmatrix} \qquad (2)$$

Thus from (1) and (2), $AB \neq BA$

Note: Matrix multiplication is not commutative in general

Example 2: If $= \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$,then find AA^t and (A^t).

Solution : Taking transpose of A, we have

$$A^t = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}, \text{so}$$

$$AA^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+9+0 & 2+0+12+0 & -6-5+6+0 \\ 2+0+12+0 & 1+0+16+4 & -3+0+8+2 \\ -6-5+6+0 & -3+0+8+2 & 9+25+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 14 & -5 \\ 14 & 21 & 7 \\ -5 & 7 & 39 \end{bmatrix}$$

As $A^t = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$, so $\left(A^t\right)^t \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$ which is A,

That is, $(A^t)^t = A$. (Note that this rule holds for any matrix A.)

Exercise 3.2

1. If $A = [a_{ij}]_{3\times4}$, then show that
i) $I_3A = A$ ii) $AI_4 = A$

2. Find the inverses of the following matrices.

i) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$ iv) $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

3. Solve the following system of linear equations.

i) $\left.\begin{matrix} 2x_1 - 3x_2 = 5 \\ 5x_1 + x_2 = 4 \end{matrix}\right\}$ ii) $\left.\begin{matrix} 4x_1 + 3x_2 = 5 \\ 3x_1 - x_2 = 7 \end{matrix}\right\}$ iii) $\left.\begin{matrix} 3x_1 - 5y = 1 \\ -2x + y = -3 \end{matrix}\right\}$

4. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & -5 \\ -1 & 0 & 4 \end{bmatrix}$, $B \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$, then find

i) $A - B$ ii) $B - A$ iii) $(A - B) - C$ iv) $A - (B - C)$

5. If $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$ and $C = \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$, then show that

i) $(AB)C = A(BC)$ ii) $(A + B)C = AC + BC$

6. If A and B are square matrices of the same order, then explain why in general;
i) $(A + B)^2 \neq A^2 + 2AB + B^2$ ii) $(A - B)^2 \neq A^2 - 2AB + B^2$
iii) $(A + B)(A - B) \neq A^2 - B^2$

7. If $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$ then find AA^t and A^tA.

8. Solve the following matrix equations for X:

i) $3X - 2A = B$ if $A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}$ and $B \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

ii) $2X - 3A = B$ if $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$ and $B \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

9. Solve the following matrix equations for A:

(i) $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$ (ii) $A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$

3.6 Determinants

The determinants of square matrices of order $n \geq 3$, can be written by following the same pattern as already discussed. For example, if $n = 4$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \text{ then the determinat of } A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Now our aim is to compute the determinants of various orders. But before describing a method for computation o f determinants of order $n \geq 3$, we introduce the following definitions.

3.6.1 Minor and Cofactor of an Element of a Matrix or its Determinant

Minor of an Element: Let us consider a square matrix A of order 3 .Then the minor of an element a_{ij} denoted by .M_{ij} is the determinant of the $(3-1)\times(3-1)$ matrix formed by deleting the ith row and the jth column of A(or|A|).
For example, if

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the matrix obtained by deleting the first row and the second column of A is $\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ (see adjoining figure) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and its determinant is the minor of an, that is,

$$M_{12}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Cofactor of an Element: The cofactor of an element a_{ij} denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} \times M_{ij}$
where M_{ij} is the minor of the element a_{ij} of A or $|A|$.

For example, $A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

3.6.2 Determinant of a Square Matrix of Order $n\geq 3$:

The determinant of a square matrix of order n is the sum of the products of each elem ent of row (or column) and its cofactor.

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$, then

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + ... + a_{in}A_{in}$ for $i = 1, 2, 3,...., n$
or $|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + ... + a_{nj}A_{nj}$ for $j = 1, 2, 3,...., n$
Putting $i = 1$, we have
$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + ... + a_{1n}A_{1n}$ which is called the expansion of |A|by the first row.

If A is a matrix of order 3, that is, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + ... + a_{in}A_{in}$ for $i = 1, 2, 3$ (1)
or $|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + ... + a_{nj}A_{nj}$ for $j = 1, 2, 3$ (2)
For example, for $i = 1$, $j = 1$ and $j = 2$, we have
$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ (i)
or $|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$ (ii)
or $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$ (iii)
(iii) can be written as: $|A| = a_{12}(-1)^{1+2} M_{12} + a_{22}(-1)^{2+2} M_{22} + a_{32}(-1)^{3+2} M_{32}$
i.e., $|A| = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$ (iv)
Similarly (i) can be written as $|A| = a_{11}M_{11} - a_{12}M_{12} - a_{13}M_{13}$ (v)
Putting the values of $M_{11}.M_{12}$: and M_{13} in (iv). we obtain

$$|A|=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or $|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$ (vi)
or $|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$ (vi)'

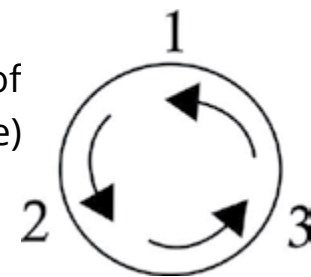
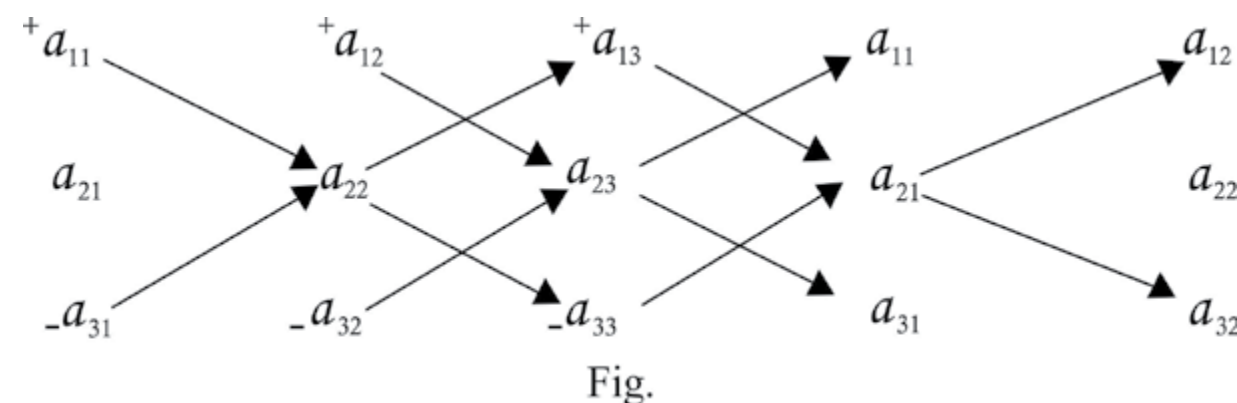
The second scripts of positive terms are in circular order of anti-clockwise direction i.e., these are as 123, 231, 312 (adjoining figure) while the second scripts of negative terms are such as 132, 213, 321.

An alternative way to remember the expansion of the determinant $|A|$ given in (vi)' is shown in the figure below.

Fig.

Example 1: Evaluate the determinant of $A=\begin{bmatrix}1 & -2 & 3\\ -2 & 3 & 1\\ 4 & -3 & 2\end{bmatrix}$

Solution : $|A|=\begin{vmatrix}1 & -2 & 3\\ -2 & 3 & 1\\ 4 & -3 & 2\end{vmatrix}$

Using the result (v) of the Art.3.6.2, that is,

$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$, we get,

$$|A|=1\begin{vmatrix}3 & 1\\ -3 & 2\end{vmatrix}-(-2)\begin{vmatrix}-2 & 1\\ 4 & 2\end{vmatrix}+3\begin{vmatrix}-2 & 3\\ 4 & -3\end{vmatrix}$$

$= 1[6 - 1(-3)] + 2[(-2).2 - 1.4] + 3[(-2)(-3) - 12]$
$= (6 + 3) + 2(-4 - 4) + 3(6 - 12)$
$= 9 - 16 - 18 = -25$

Example 2: Find the cofactors A_{12}, A_{22} and A_{32} if $A=\begin{bmatrix}1 & -2 & 3\\ -2 & 3 & 1\\ 4 & -3 & 2\end{bmatrix}$ and find $|A|$.

Solution : We first find M_{12}, M_{22} and M_{32},

$$M_{12}=\begin{vmatrix}-2 & 1\\ 4 & 2\end{vmatrix} = -4-4=-8\,; \quad M_{22}=\begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix} = 2-12=-10 \text{ and } M_{32}=\begin{vmatrix}1 & 3\\ -2 & 1\end{vmatrix} = 1-(-6)=7$$

Thus $A_{12} = (-1)^{1+2}M_{12} = (-1)(-8) = 8$; $A_{22} = (-1)^{2+2}M_{22} = 1(-10) = -10$
$A_{32} = (-1)^{3+2}M_{32} = (-1)(7) = -7$;
and $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = (-2)8 + 3(-10) + (-3)(-7)$
$= -16 - 30 + 21 = -25$
Note that $a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 1(8) + (-2)(-10) + 4(-7)$
$= 8 + 20 - 28 = 0$
and $a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32} = 3(8) + 1(-10) + 2(-7)$
$= 24 - 10 - 14 = 0$
Similarly we can show that $a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33} = 0$;
$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$; and $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$;

3.7 Properties of Determinants which Help in their Evaluation

1. For a square matrix A, $|A| = |A^t|$
2. If in a square matrix A, two rows or two columns are interchanged, the determinant of the resulting matrix is $-|A|$.
3. If a square matrix A has two identical rows or two identical columns, then $|A| = 0$.
4. If all the entries of a row (or a column) of a square matrix A are zero, then $|A| = 0$.
5. If the entries of a row (or a column) in a square matrix A are multiplied by a number $k \in \Re$, then the determinant of the resulting matrix is $k\,|A|$.
6. If each entry of a row (or a column) of a square matrix consists of two terms, then its determinant can be written as the sum of two determinants, i.e., if

$$B=\begin{bmatrix}a_{11}+b_{11} & a_{12} & a_{13}\\ a_{21}+b_{21} & a_{22} & a_{23}\\ a_{31}+b_{31} & a_{32} & a_{33}\end{bmatrix}, \quad \text{then}$$

$$|B| = \begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21} & b_{21} = a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & +a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

7. If to each entry of a row (or a column) of a square matrix A is added a non-zero multiple of the corresponding entry of another row (or column), then the determinant of the resulting matrix is $|A|$.
8. If a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

Now we prove the above mentioned properties of determinants.

Proporty 1: If the rows and columns of a determinant are interchanged, then the value of the determinant does not change. For example.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - a_{21}a_{12} = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \text{ (rows and columns are interchanged)}$$

Property 2: The value of a determinant changes sign if any two rows (columns) are interchanged. For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

and $\begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix} = a_{12}a_{21} - a_{11}a_{22} = -(a_{11}a_{22} - a_{12}a_{21})$ (columns are interchanged)

Property 3: If all the entries in any row (column) are zero, the value of the determinant is zero. For example,

$$\begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = 0\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - 0\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + 0\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = 0 \quad \text{(expanding by C)}$$

Property 4: If any two rows (columns) of a determinant are identical, the value of the determinant is zero. For example,

$$\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = 0, \quad \text{(it can be proved by expanding the determinant)}$$

Property 5: If any row (column) of a determinant is multiplied by a non-zero number k, the value of the new determinant becomes equal to k times the value of original determinant. For example,

$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, multiplying first row by a non-zero number k, we get

$$\begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = ka_{11}a_{22} - ka_{12}a_{21} = k(a_{11}a_{22} - a_{12}a_{21}) = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Property 6: If any row (column) of a determinant consists of two terms, it can be written as the sum of two determinants as given below:

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{(proof is left for the reader)}$$

Property 7: If any row (column) o f a determinant is multiplied by a non-zero number k and the result is added to the corresponding entries of another row (column), the value of the determinant does not change. For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12}+ka_{11} \\ a_{21} & a_{22}+ka_{21} \end{vmatrix} \quad (k \text{ multiple of } C_1 \text{ is added to } C_2)$$

It can be proved by expanding both the sides. Proof is left for the reader.

Example 3: If $A = \begin{bmatrix} 2 & -2 & 3 & 4 \\ 3 & 1 & 5 & -1 \\ -5 & -3 & 1 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}$, evaluate $|A|$

Solution : $$|A| = \begin{vmatrix} 2 & -2 & 3 & 4 \\ 3 & 1 & 5 & -1 \\ -5 & -3 & 1 & 0 \\ 1 & -1 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 3 & 0 \\ 0 & 4 & 5 & -7 \\ 0 & -8 & 1 & 10 \\ 1 & -1 & 0 & 2 \end{vmatrix} \quad \text{By } R_1 + (-2)R_4,\ R_2 + (-3)R_4 \text{ and } R_3 + 5R_4$$

Expanding by first column, we have

$|A| = 0.A_{11} + 0.A_{21} + 0.A_{31} + 1.A_{41}$

$$= (-1)^{4+1} \times \begin{vmatrix} 0 & 3 & 0 \\ 4 & 5 & -7 \\ -8 & 1 & 10 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 3 & 0 \\ 4 & 5 & -7 \\ -8 & 1 & 10 \end{vmatrix}$$

$= (-1)(-3)[4 \times 10 - (-7)(-8)] = 3(40 - 56) = -48$

Example 4: Without expansion, show that $\begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix} = 0$

Solution : Multiplying each entry of C_1 by –1 and adding to the corresponding entry of C_2 i.e., by $C_2 + (-1)C_1$, we get

$$\begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix} = \begin{vmatrix} x & a+x+(-1)x & b+c \\ x & b+x+(-1)x & c+a \\ x & c+x+(-1)x & a+b \end{vmatrix}$$

$$= \begin{vmatrix} x & a & b+c \\ x & b & c+a \\ x & c & a+b \end{vmatrix} = x\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} \begin{pmatrix} \text{by property 5 or} \\ \text{taking x common} \\ \text{from } C_1 \end{pmatrix}$$

$$= x\begin{vmatrix} 1 & a+(b+c) & b+c \\ 1 & b+(c+a) & c+a \\ 1 & c+(a+b) & a+b \end{vmatrix}, \begin{pmatrix} \text{adding the entries of } C_3 \text{ to the} \\ \text{corresponding entries of } C_2 \end{pmatrix}$$

$$= x(a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}, \text{ (by property 5)}$$

$= x(a + b + c).0$ ($\because$ C_1 and C_2 are identical or by property 3)

$= 0$

Example 5: Solve the equation $\begin{vmatrix} x & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -2 & 3 & 4 \\ -2 & x & 1 & -1 \end{vmatrix} = 0$

Solution: By $C_3 + C_2$ and $C_4 + C_2$, we have

$$\begin{vmatrix} x & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 2 \\ -2 & x & x+1 & x-1 \end{vmatrix} = 0$$

Expanding by R_2, we get $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ -2 & x+1 & x-1 \end{vmatrix} = 0$ ($\because$ $(-1)^{2+2} = 1$)

By $R_3 + 2R_2$, we get $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ 0 & x+3 & x+3 \end{vmatrix} = 0$

or $(x+3)\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 0$ (by taking $x + 3$ common from R_3)

$\Rightarrow$ $x + 3 = 0$ or $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 0$

$\Rightarrow$ $x = -3$ or $x = 0$ ($\because$ R_1 and R_3 are identical if $x = 0$)

Thus the solution set is {–3, 0}.

3.8 Adjoint and Inverse of a Square Matrix of Order $n \geq 3$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the matrix of co-factors of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$,

and $\text{adj}\, A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$,

Inverse of a Square Matrix of Order n≥3: Let A be a non singular square matrix of order n. If there exists a matrix B such that $AB = BA = I_n$, then B is called the multiplicative inverse of A and is denoted by A^{-1}. It is obvious that the order of A^{-1} is $n \times n$.

Thus $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

If A is a non singular matrix, then

$$A^{-1} = \frac{1}{|A|} \text{adj} A$$

Example 6: Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution: We first find the cofactors of the elements of A.

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 1.(2+1) = 3, \qquad A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(-1) = 1$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = 1.(0-2) = -2, \qquad A_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = (-1)(0+2) = -2$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1.(1-2) = -1, \qquad A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = (-1)(-1-0) = 1$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 1.(0-4) = -4, \qquad A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)(1-0) = -1$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 1.(2-0) = 2$$

Thus $[A_{ij}]_{3\times3} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$

and $\text{adj}\, A = [A'_{ij}]_{3\times3} = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ $(\because A'_{ij} = A_{ji}$ for $i, j = 1, 2, 3)$

Since $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$
$= 1(3) + 0(1) + 2(-2)$
$= 3 + 0 - 4 = -1,$

So $A^{-1} = \frac{1}{|A|} \text{adj}\, A = \frac{1}{-1} \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$

Example 7: If $A = \begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 2 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ then verify that

$(AB)^t = B^t A^t$

Solution: so $AB = \begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1-4 & -3+2 \\ 1-8 & 3+4 \\ 2+2 & 6-1 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -7 & 7 \\ 4 & 5 \end{bmatrix}$,

$(AB)^t = \begin{bmatrix} -5 & -7 & 4 \\ -1 & 7 & 5 \end{bmatrix}$

and $B^t A^t = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix} \left(\because A^t = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix} \textit{and } B^t = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \right)$

$= \begin{bmatrix} -1-4 & 1-8 & 2+2 \\ -3+2 & 3+4 & 6-1 \end{bmatrix} \begin{bmatrix} -5 & -7 & 4 \\ -1 & 7 & 5 \end{bmatrix}$

Thus $(AB)^t = B^t A^t$

Exercise 3.3

Evaluate the following determinants.

1. i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$ ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$ iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$ v) $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$ vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

2. Without expansion show that

i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$ ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$ iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

3. Show that

i) $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$ iii) $\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$

iv) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ v) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

vi) $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3 \quad b^3$ vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$

viii) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3+b^3+c^3-3abc$

ix) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$

x) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

xi) $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

4. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & 1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$, then find;

i) A_{12}, A_{22}, A_{32} and $|A|$ ii) B_{21}, B_{22}, B_{23} and $|B|$

5. Without expansion verify that

i) $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$ ii) $\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$ iii) $\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ca} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix} = 0$

iv) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

v) $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} = 0$

vi) $\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

vii) $\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$

viii) $\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$ ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$

6. Find values of x if

i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & -4 \\ x & 1 & 0 \end{vmatrix} = 30$ ii) $\begin{vmatrix} 1 & x-1 & 3 \\ 1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$ iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

7. Evaluate the following determinants:

i) $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$ ii) $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$ iii) $\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix}$

8. Show that $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$

9. Find $|AA^t|$ and $|A^tA|$ if

i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$ ii) $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

10. If A is a square matrix of order 3, then show that $|KA| = k^3|A|$.
11. Find the value of λ if A and B singular.

$A = \begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$

12. Which of the following matrices are singular and which of them are non singular?

i) $\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{bmatrix}$

13. Find the inverse of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ and show that $A^{-1}A = I_3$

14. Verify that $(AB)^{-1} = B^{-1}A^{-1}$ if

i) $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

15. Verify that $(AB)^t = B^t A^t$ and if

$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$

16. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ verify that $(A^{-1})^t = (A^t)^{-1}$

17. If A and B are non-singular matrices, then show that

i) $(AB)^{-1} = B^{-1}A^{-1}$ ii) $(A^{-1})^{-1} = A$

3.9 Elementary Row and Column Operations on a Matrix

Usually a given system of linear equations is reduced to a simple equivalent system by applying in turn a finite number of elementary operations which are stated as below:

1. Interchanging two equations.
2. Multiplying an equation by a non-zero number.
3. Adding a multiple of one equation to another equation.

Note: The systems of linear equations involving the same variables, are equivalent if they have the same solution.

Corresponding to these three elementary operations, the following elementary row operations are applied to matrices to obtain equivalent matrices.

i) Interchanging two rows
ii) Multiplying a row by a non-zero number
iii) Adding a multiple of one row to another row.

Note: Matrices A and B are equivalent if B can be obtained by applying in turn a finite number of row operations on A.

Notations that are used to represent row operations for I to III are given below:
Interchanging R_i and R_j is expressed as $R_i \leftrightarrow R_j$.
k times R_i, is denoted by $kR_i \rightarrow R'_i$
Adding k times R_j to R_i is expressed as $R_i + kR_j \rightarrow R'_i$
(R'_i is the new row obtained after applying the row operation).

For equivalent matrices A and B, we write $A \underset{\sim}{R} B$.

If $A \underset{\sim}{R} B$ then $B \underset{\sim}{R} A$. Also if $A \underset{\sim}{R} B$ and $B \underset{\sim}{R} C$, then $A \underset{\sim}{R} C$. Now we state the elementary column operations and notations that are used for them.

i) Interchanging two columns $C_i \leftrightarrow C_j$
ii) Multiplying a column by a non-zero number $kC_i \rightarrow C'_i$
iii) Adding a multiple of one column to another column $C_i + kC_j \rightarrow C'_i$

Consider the system of linear equations;

$$\left.\begin{aligned} x + y + 2z &= 1 \\ 2x - y = 8z &= 12 \\ 3x + 5y + 4z &= 3 \end{aligned}\right\}$$ which can be written in matrix forms as

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 8 \\ 3 & 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \\ -3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 5 \\ 2 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 12 & -3 \end{bmatrix}$$

that is, $AX = B$ (i) $X^t A^t = B^t$ (ii)

where $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 8 \\ 3 & 5 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 12 \\ -3 \end{bmatrix}$

A is called the matrix of coefficients.

Appending a column of constants on the left of A, we get the augmented matrix of the given system, that is,

$$\begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 2 & -1 & 8 & \vdots & 12 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix}$$ (Appended column is separated by a dotted line segment)

Now we explain the application of elementary operations on the system of linear equations and the application of elementary row operations on the augmented matrix of the system writing them side by side.

$$\left.\begin{aligned} x + y + 2z &= 1 \\ 2x + -y + 8z &= 12 \\ 3x + 5y + 4z &= 3 \end{aligned}\right\} \qquad \begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 2 & 1 & 8 & \vdots & 12 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix}$$

Adding –2 times the first equation to the second and –3 times the first equation to the third, we get (By $R_2 + (-2)R_1 \rightarrow R'_2$ and $R_3 + (-3)R_1 \rightarrow R'_3$, we get)

$$\left.\begin{aligned} x + y + 2z &= 1 \\ -3y + 4z &= 10 \\ 2y - 2z &= 6 \end{aligned}\right\} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 0 & 3 & 4 & \vdots & 10 \\ 0 & 2 & -2 & \vdots & -6 \end{bmatrix}$$

Interchanging the second and third equations, we have (By $R_2 \leftrightarrow R_3$, we get)

$$\left.\begin{aligned} x+y+2z&=1\\ 2y-2z&=6\\ -3y+4z&=10 \end{aligned}\right\} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1\\ 0 & 2 & -2 & \vdots & -6\\ 0 & 3 & 4 & \vdots & 10 \end{bmatrix}$$

Multiplying the second equation by $\frac{1}{2}$, we get By $\frac{1}{2}R_2 \rightarrow R'_2$, we get.

$$\left.\begin{aligned} x+y+2z&=1\\ y-z&=-3\\ -3y+4z&=10 \end{aligned}\right\} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1\\ 0 & 1 & -1 & \vdots & -3\\ 0 & -3 & 4 & \vdots & 10 \end{bmatrix}$$

Adding 3 times the second equation to By $R_3 + 3R_2 \rightarrow R'_3$, we obtain, the third, we obtain,

$$\left.\begin{aligned} x+y+2z&=1\\ y-z&=-3\\ z&=1 \end{aligned}\right\} \qquad \underset{\sim}{R}\begin{bmatrix} 1 & 1 & 2 & \vdots & 1\\ 0 & 1 & -1 & \vdots & -3\\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

The given system is reduced to the triangular form which is so called because on the left the coefficients (of the terms) within the dotted triangle are zero.

Putting $z = 1$ in $y - z = -3$, we have $y - 1 = -3 \Rightarrow y = -2$

Substiliting $z = 1$, $y = -2$ in the first equation, we get

$$x + (-2) + 2(1) = 1 \Rightarrow x = 1$$

Thus the solution set o f the given system is $\{(1, -2, 1)\}$.

Appending a row of constants below the matrix A^t, we obtain the augmented matrix for the matrix equation (ii), that is

$$\begin{bmatrix} 1 & 2 & 3\\ 1 & -1 & 5\\ 2 & 8 & 4\\ \cdots & \cdots & \cdots\\ 1 & 12 & -3 \end{bmatrix}$$

Now we apply elementary column operations to this augmented matrix.

$$\begin{bmatrix} 1 & 2 & 3\\ 1 & -1 & 5\\ 2 & 8 & 4\\ \cdots & \cdots & \cdots\\ 1 & 12 & -3 \end{bmatrix} \underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0\\ 1 & -3 & 2\\ 2 & 4 & -2\\ \cdots & \cdots & \cdots\\ 1 & 10 & -6 \end{bmatrix}$$ By $C_2 + (-2)C_1 \rightarrow C'_2$ and $C_3 + (-3)C_1 \rightarrow C'_3$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0\\ 1 & 2 & -3\\ 2 & -2 & 4\\ \cdots & \cdots & \cdots\\ 1 & -6 & 10 \end{bmatrix} \text{By } C_2 \leftrightarrow C_3 \qquad \underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & -3\\ 2 & -1 & 4\\ \cdots & \cdots & \cdots\\ 1 & -3 & 10 \end{bmatrix} \text{By } \frac{1}{2}C_2 \rightarrow C'_2$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 2 & -1 & 1\\ \cdots & \cdots & \cdots\\ 1 & -3 & 1 \end{bmatrix} \text{By } C_3 + 3C_2 \rightarrow C'_3$$

Thus $\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \end{bmatrix}$

or $\begin{bmatrix} x+y+2z & y-z & z \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \end{bmatrix}$

$$\Rightarrow \quad \left.\begin{aligned} x+y+2z&=1\\ y-z&=-3\\ z&=1 \end{aligned}\right\}$$

Upper Triangular Matrix: A square matrix $A = [a_{ij}]$ is called upper triangular if all elements below the principal diagonal are zero, that is,

$$a_{ij} = 0 \text{ for all } i > j$$

Lower Triangular Matrix: A square matrix $A = [a_{ij}]$ is said to be lower triangular if all elements above the principal diagonal are zero, that is,

$$a_{ij} = 0 \text{ for all } i < j$$

Triangular Matrix: A square matrix A is named as triangular whether it is upper triangular or lower triangular. For example, the matrices

$$\begin{bmatrix}1&2&3\\0&1&4\\0&0&6\end{bmatrix} \text{ and } \begin{bmatrix}1&0&0&0\\3&2&0&0\\4&1&5&0\\-1&2&3&1\end{bmatrix}$$ are triangular matrices of order 3 and 4

respectively. The first matrix is upper triangular while the second is lower triangular.

Note: Diagonal matrices are both upper triangular and lower triangular.

Symmetric Matrix: A square matrices $A = [a_{ij}]_{n\times n}$ is called symmetric if $A^t = A$.
From $A^t = A$, it follows that $[a'_{ij}]_{n\times n} = [a_{ij}]_{n\times n}$
which implies that $a'_{ij} = a_{ij}$ for $i, j = 1, 2, 3, \ldots\ldots, n$.
but by the definition of transpose, $a'_{ij} = a_{ji}$ for $i, j = 1, 2, 3, \ldots\ldots, n$.
Thus $a_{ij} = a_{ji}$ for $i, j = 1, 2, 3, \ldots\ldots, n$.
and we conclude that a square matrix $A = [a_{ij}]_{n\times n}$ is symmetric if $a_{ij} = a_{ji}$.
For example, the matrices

$$\begin{bmatrix}1&3\\3&2\end{bmatrix}, \begin{bmatrix}a&h&g\\h&b&f\\g&f&c\end{bmatrix} \text{ and } \begin{bmatrix}1&3&2&-1\\3&0&5&6\\2&5&1&-2\\-1&6&-2&3\end{bmatrix}$$ are symmetric.

Skew Symmetric Matrix : A square matrix A = $[a_{ij}]_{n\times n}$ is called skew symmetric or anti-symmetric if $A^t = -A$.

From $A^t = -A$, it follows that $[a'_{ij}]$ = for $i, j = 1, 2, 3, \ldots\ldots, n$
which implies that $a'_{ij} = -a_{ij}$ for $i, j = 1, 2, 3, \ldots\ldots, n$
but by the definition of transpose $a'_{ij} = a_{ji}$ for $i, j = 1, 2, 3, \ldots, n$
Thus $-a_{ij} = a_{ji}$ or $a_{ij} = -a_{ji}$

Alternatively we can say that a square matrix $A = [a_{ij}]_{nxn}$ is anti-symmetric if $a_{ij} = -a_{ji}$.

For diagonal elements $j = i$, so
$a_{ii} = -a_{ii}$ *or* $2a_{ii} = 0 \Rightarrow$ $a_{ii} = 0$ for $i = 1, 2, 3, \ldots\ldots, n$

For example if B = $\begin{bmatrix}0&-4&1\\4&0&-3\\-1&3&0\end{bmatrix}$, then

$$B^t = \begin{bmatrix}0&4&-1\\-4&0&3\\1&-3&0\end{bmatrix} = (-1)\begin{bmatrix}0&-4&1\\4&0&-3\\-1&3&0\end{bmatrix} = -B$$

Thus the matrix B is skew-symmetric.

Let $A = [a_{ij}]$ be an n x m matrix with complex entries, Then the n x m matrix $[\bar{a}_{ij}]$ where $\bar{a}_{ij}$ is the complex conjugate of a_{ij} for all i, j, is called *conjugate* of A and is denoted by $\bar{A}$. For example, if

$$A = \begin{bmatrix}3-i&-i\\2i&1+i\end{bmatrix}, \text{then } \bar{A} \begin{bmatrix}\overline{3-i}&\overline{-i}\\\overline{2i}&\overline{1+i}\end{bmatrix} \begin{bmatrix}3+i&i\\-2i&1-i\end{bmatrix}$$

Hermitian Matrix: A square matrix $A = [a_{ij}]_{nxn}$ with complex entries, is called *hermitian* if $\left(\bar{A}\right)^t = A$.

From, $\left(\bar{A}\right)^t = A$ it follows that $\left[\bar{a}'_{ij}\right]_{nxn} = [a_{ij}]_{nxn}$ which implies that $\bar{a}'_{ij} = a_{ij}$ for $i, j = 1, 2, 3, \ldots, n$ but by the definition of transpose, $\bar{a}'_{ij} = \bar{a}_{ij}$ for $i, j = 1, 2, 3, \ldots\ldots, n$.

Thus $a_{ij} = \bar{a}_{ij}$ for $i, j = 1, 2, 3, \ldots\ldots, n$ and we can say that a square matrix $A = [a_{ij}]_{nxn}$ is *hermitian* if $a_{ij} = \bar{a}_{ji}$ for $i, j = 1, 2, 3, \ldots, n$.

For diagnal elements, $j = i$ so $a_{ii} = \bar{a}_{ii}$ which implies that a_{ii} is real for $i = 1, 2, 3, \ldots, n$

For example, if A = $\begin{bmatrix}1&1-i\\1+i&2\end{bmatrix}$, then

$$\bar{A} = \begin{bmatrix}1&1+i\\1-i&2\end{bmatrix} \Rightarrow \left(\bar{A}\right)^t = \begin{bmatrix}1&1-i\\1+i&2\end{bmatrix} = A$$

Thus A is hermitian.

Skew Hermitian Matrix: A square matrix $A = [a_{ij}]_{nxn}$ with complex entries, is called skew-hermitian or anti-hermitian if $(\bar{A})^t = -A$.

From $(\bar{A})^t = -A$, it follows that $\left[\bar{a}_{ij}^{\,t}\right]_{n\times n} = [-a_{ij}]_{nxn}$

which implies that $\bar{a}_{ij}^{\,t} = -a_{ij}$ for $i, j = 1, 2, 3, ..., n$.

but by the definition of transpose, $\bar{a}_{ij}^{\,t} = \bar{a}_{ji}$ for $i, j = 1, 2, 3, ..., n$.

Thus $-a_{ij} = \bar{a}_{ji}$ or $a_{ij} = -\bar{a}_{ji}$, for $i, j = 1, 2, 3, ..., n$.

and we can conclude that a square matrix $A = [a_{ij}]_{nxn}$ is anti-hermitian if $a_{ij} = -\bar{a}_{ji}$

For diagonal elements $j = i$, so $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + -\bar{a}_{ii} = 0$

which holds if $a_{ii} = 0$ or $a_{ii} = i\lambda$ where λ is real

because $0 + 0 = 0$ or $i\lambda + i\bar{\lambda} = i\lambda - i\lambda = 0$

For example, if $A = \begin{bmatrix} 0 & 2+3i \\ -2+3i & 0 \end{bmatrix}$, then

$$\bar{A} = \begin{bmatrix} 0 & 2+3i \\ -2+3i & 0 \end{bmatrix}$$

$$\Rightarrow (\bar{A})^t = \begin{bmatrix} 0 & -2+3i \\ 2+3i & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & 2-3i \\ -2-3i & 0 \end{bmatrix} = -A$$

Thus A is skew-hermitian.

3.10 Echelon and Reduced Echelon Forms of Matrices

In any non-zero row of a matrix, the first non-zero entry is called the *leading entry* of that row. The zeros before the leading entry of a row are named as the *leading zero entries* of the row.

Echelon Form of a Matrix: An m x n matrix A is called in **(row) echelon form** if

i) In each successive non-zero row, the number of zeros before the leading entry is greater than the number of such zeros in the preceding row,

ii) Every non-zero row in A precedes every zero row (if any),

iii) The first non-zero entry (or leading entry) in each row is 1.

Note: Some authors do not require the condition (iii)

The matrices $\begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in echelon form

but the matrices $\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 4 \end{bmatrix}$ are not in echelon form.

Reduced Echelon Form of a Matrix: An m x n matrix A is said to be in **reduced (row) echelon form** if it is in (row) echelon form and if the first non-zero entry (or leading entry) in R_i lies in C_j, then all other entries of C_j are zero.

The matrices $\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in (row) reduced echelon form.

Example 1: Reduce the following matrix to (row) echelon and reduced (row) echelon form,

$$\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$$

Solution: $\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ $\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 3 & -1 & 9 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ By $R_1 \leftrightarrow R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -5 & 15 \\ 0 & 4 & -3 & 11 \end{bmatrix}$ By $R_2 + (-2)R_1 \to R'_2$ and $R_3 + (-3)R_1 \to R'_3$ $\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 4 & -3 & 11 \end{bmatrix}$ By $\frac{1}{5}R_2 \to R'_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -1 \end{bmatrix}$ By $R_3 + (-4)R_2 \to R'_3$ $\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ By $R_1 + 1.R_2 \to R'_1$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1\end{bmatrix} \quad \begin{matrix}\text{By } R_1+(-1)R_3 \rightarrow R'_1\\ \text{and } R_2+1.R_3 \rightarrow R'_2\end{matrix}$$

Thus $\begin{bmatrix}1 & -1 & 2 & 3\\ 0 & 1 & -1 & 3\\ 0 & 0 & 1 & -1\end{bmatrix}$ and $\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1\end{bmatrix}$ are (row) echelon and reduced (row) echelon forms of the given matrix respectively.

Let *A* be a non-singular matrix. If the application of elementary row operations on $A \vdots I$ in succession reduces *A* to I, then the resulting matrix is $I \vdots A^{-1}$.

Similarly if the application of elementary column operations on $\begin{matrix}A\\ \cdots\\ I\end{matrix}$ in succession reduces *A* to I, then the resulting matrix is $\dfrac{I}{A^{-1}}$

Thus $A \vdots I \ \underset{\sim}{R} \ I \vdots A^{-1}$ and $\begin{matrix}A\\ \cdots\\ I\end{matrix} \ \underset{\sim}{C} \ \begin{matrix}I\\ \cdots\\ A^{-1}\end{matrix}$

Example 2: Find the inverse of the matrix A = $\begin{bmatrix}2 & 5 & -1\\ 3 & 4 & 2\\ 1 & 2 & -2\end{bmatrix}$

Solution: $|A| = \begin{vmatrix}2 & 5 & -1\\ 3 & 4 & 2\\ 1 & 2 & -2\end{vmatrix}$ = 2(–8 – 4) – 5(–6 – 2) –1(6 – 4) = –24 + 40 – 2

= 40 – 26 = 14 As |*A*| ≠ 0, so *A* is non-singular.

Appending I_3 on the left of the matrix *A*, we have $\begin{bmatrix}2 & 5 & -1 & \vdots & 1 & 0 & 0\\ 3 & 4 & 2 & \vdots & 0 & 1 & 0\\ 1 & 2 & -2 & \vdots & 0 & 0 & 1\end{bmatrix}$

Interchanging R_1 and R_3 we get..

$$\begin{bmatrix}1 & 2 & -2 & \vdots & 0 & 0 & 1\\ 3 & 4 & 2 & \vdots & 0 & 1 & 0\\ 2 & 5 & -1 & \vdots & 1 & 0 & 0\end{bmatrix} \underset{\sim}{R} \begin{bmatrix}1 & 2 & -2 & \vdots & 0 & 0 & 1\\ 0 & -2 & 8 & \vdots & 0 & 1 & -3\\ 0 & 1 & 3 & \vdots & 1 & 0 & -2\end{bmatrix} \begin{matrix}\text{By } R_2+(-3)R_1 \rightarrow R'_2\\ \text{and } R_3+(-2)R_1 \rightarrow R'_3\end{matrix}$$

By $-\frac{1}{2}R_2 \rightarrow R'_2$, we get

$$\begin{bmatrix}1 & 2 & -2 & \vdots & 0 & 0 & 1\\ 0 & 1 & -4 & \vdots & 0 & -\frac{1}{2} & \frac{3}{2}\\ 0 & 1 & 3 & \vdots & 1 & 0 & -2\end{bmatrix} \underset{\sim}{R} \begin{bmatrix}1 & 0 & 6 & \vdots & 0 & 0 & -2\\ 0 & 1 & -4 & \vdots & 0 & -\frac{1}{2} & \frac{3}{2}\\ 0 & 0 & 7 & \vdots & 1 & \frac{1}{2} & -\frac{7}{2}\end{bmatrix} \begin{matrix}\text{By } R_3+(-1)R_2 \rightarrow R'_3\\ \text{and } R_1+(-2)R_2 \rightarrow R'_1\end{matrix}$$

By $\frac{1}{7}R_3 \rightarrow R'_3$, we have

$$\begin{bmatrix}1 & 0 & 6 & \vdots & 0 & 1 & -2\\ 0 & 1 & -4 & \vdots & 0 & -\frac{1}{2} & \frac{3}{2}\\ 0 & 0 & 1 & \vdots & \frac{1}{7} & \frac{1}{14} & -\frac{1}{2}\end{bmatrix} \underset{\sim}{R} \begin{bmatrix}1 & 0 & 0 & \vdots & -\frac{6}{7} & \frac{4}{7} & 1\\ 0 & 1 & 0 & \vdots & \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2}\\ 0 & 0 & 1 & \vdots & \frac{1}{7} & \frac{1}{14} & -\frac{1}{2}\end{bmatrix} \begin{matrix}\text{By } R_1+(-6)R_3 \rightarrow R'_1\\ \text{and } R_2+4R_3 \rightarrow R'_2\end{matrix}$$

Thus the inverse of *A* is $\begin{bmatrix}-\frac{6}{7} & \frac{4}{7} & 1\\ -\frac{4}{7} & -\frac{3}{14} & -\frac{1}{2}\\ \frac{1}{7} & \frac{1}{14} & -\frac{1}{2}\end{bmatrix}$

Appending I_3 below the matrices *A*, we have

$$\begin{bmatrix}2 & 5 & -1\\ 3 & 4 & 2\\ 1 & 2 & -2\\ \cdots & \cdots & \cdots\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Interchanging C_1 and C_3, we get

$$\begin{bmatrix} 2 & 5 & -1 \\ 3 & 4 & 2 \\ 1 & 2 & -2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{C} \begin{bmatrix} -1 & 5 & 2 \\ 2 & 4 & 3 \\ -2 & 2 & 1 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \underset{\sim}{C} \begin{bmatrix} 1 & 5 & 2 \\ -2 & 4 & 3 \\ 2 & 2 & 1 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \text{ By } (-1)C_1 \rightarrow C'_1$$

By $C_2 + (-5)C_1 \rightarrow C'_2$ and $C_3 + (-2)C_1 \rightarrow C'_3$, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 14 & 7 \\ 2 & -8 & -3 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 5 & 2 \end{bmatrix} \underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 7 \\ 2 & -\frac{4}{7} & -3 \\ \hline 0 & 0 & 1 \\ 0 & \frac{1}{14} & 0 \\ -1 & \frac{5}{14} & 2 \end{bmatrix} \quad \text{By } \frac{1}{14}C_2 \rightarrow C'_2$$

By $C_1 + (2)C_2 \rightarrow C'_1$ and $C_3 + (-7)C_2 \rightarrow C'_3$ we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{6}{7} & -\frac{4}{7} & 1 \\ \hline 0 & 0 & 1 \\ \frac{1}{7} & \frac{1}{14} & \frac{1}{2} \\ -\frac{2}{7} & \frac{5}{14} & -\frac{1}{2} \end{bmatrix} \underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -\frac{6}{7} & \frac{4}{7} & 1 \\ \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix} \begin{array}{l} \text{By } C_1 + \left(-\frac{6}{7}\right)C_3 \rightarrow C'_1 \\ \text{and } C_2 + \left(\frac{4}{7}\right)C_3 \rightarrow C'_2 \end{array}$$

Thus the inverse of A is $\begin{bmatrix} -\frac{6}{7} & \frac{4}{7} & 1 \\ \frac{4}{7} & -\frac{3}{14} & -\frac{1}{2} \\ \frac{1}{7} & \frac{1}{14} & -\frac{1}{2} \end{bmatrix}$

Rank of a Matrix: Let A be a non-zero matrix. If r is the number of non-zero rows when it is reduced to the reduced echelon form, then r is called the (row) **rank** of the matrix A.

Example 3: Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$

Solution: $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 3 & -1 \\ 0 & 4 & 6 & -2 \end{bmatrix} \begin{array}{l} \text{By } R_2 + (-2)R_1 \rightarrow R'_2 \\ \text{and } R_3 + (-3)R_1 \rightarrow R'_3 \end{array}$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 4 & 6 & -2 \end{bmatrix} \text{By } \frac{1}{2}R_2 \rightarrow R'_2 \quad \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{By } R_3 + (-4)R_2 \rightarrow R'_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & \frac{7}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{By } R_1 + 1.R_2 \rightarrow R'_1$$

As the number of non-zero rows is 2 when the given matrix is reduced to the reduced echelon form, therefore, the rank of the given matrix is 2.

Exercise 3.4

1. If $A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$, then show that $A + B$ is symmetric.

2. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$, show that

i) $A + A^t$ is symmetric   ii) $A - A^t$ is skew-symmetric.

3. If A is any square matrix of order 3, show that

i) $A + A^t$ is symmetric and   ii) $A - A^t$ is skew-symmetric.

4. If the matrices A and B are symmetric and $AB = BA$, show that AB is symmetric.

5. Show that AA^t and A^tA are symmetric for any matrix of order 2 x 3.

6. If $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$, show that

i) $A + \left(\overline{A}\right)^t$ is hermitian   ii) $A - \left(\overline{A}\right)^t$ is skew-hermitian.

7. If A is symmetric or skew-symmetric, show that A^2 is symmetric.

8. If $A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$, find $A\left(\overline{A}\right)^t$.

9. Find the inverses of the following matrices. Also find their inverses by using row and column operations.

i) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

10. Find the rank of the following matrices

i) $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$

3.11 System of Linear Equations

An equation of the form:

$$ax + by = k \qquad \text{(i)}$$

where $a \neq 0, b \neq 0, k \neq 0$

is called a non-homogeneous linear equation in two variables x and y.

Two linear equations in the same two variables such as:

$$\left.\begin{aligned} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{aligned}\right\} \qquad \text{(I)}$$

is called a system of non-homogeneous linear equations in the two variables x and y if constant terms k_1, k_2 are not both zero.

If in the equation (i), $k = 0$, that is, $ax + by = 0$, then it is called a homogeneous linear equation in x and y.

If in the system (I), $k_1 = k_2 = 0$, then it is said to be a system of homogenous linear equations in x and y.

An equation of the form:

$$ax + by + cz = k \qquad \text{.... (ii)}$$

is called a non-homogeneous linear equation in three variables x, y and z if $a \neq 0, b \neq 0, c \neq 0$ and $k \neq 0$. Three linear equations in three variables such as:

$$\left.\begin{aligned} a_1x + b_1y + c_1z = k_1 \\ a_2x + b_2y + c_2z = k_2 \\ a_3x + b_3y + c_3z = k_3 \end{aligned}\right\} \qquad \text{(II)}$$

is called a system of non-homogeneous linear equations in the three variables x, y and z, if constant terms k_1, k_2 and k_3 are not all zero.

If in the equations (ii) $k = 0$ that is, $ax + by + cz = 0$.

then it is called a homogeneous linear equation it x, y and z.

If in the system (II), $k_1 = k_2 = k_3 = 0$, then it is said to be a system of homogeneous linear equations in x ; y and z.

A system of linear equations is said to be consistent if the system has a unique solution or it has infinitely many solutions.

A system of linear equations is said to be inconsistent if the system has no solution.

The system (II), consists of three equations in three variables so it is called 3 x 3 linear system but a system of the form:

$$\left.\begin{array}{l} x - y + 2z = 6 \\ 2x + y + 3z = 4 \end{array}\right\}$$

is named as 2 x 3 linear system.

Now we solve the following three 3 x 3 linear systems to determine the criterion for a system to be consistent or for a system to be inconsistent.

$$\left.\begin{array}{r} 2x + 5y - z = 5 \\ 3x + 4y + 2z = 11 \\ x + 2y - 2z = 3 \end{array}\right\} \quad ....(1), \qquad \left.\begin{array}{r} x + y + 2z = 1 \\ 2x - y + 7z = 11 \\ 3x + 5y + 4z = 3 \end{array}\right\} \quad (2)$$

and
$$\left.\begin{array}{r} x - y + 2z = 1 \\ 2x - 6y + 5z = 7 \\ 3x + 5y + 4z = 3 \end{array}\right\} \quad ....(3)$$

The augmented matrix of the system (1) is

$$\begin{bmatrix} 2 & 5 & -1 & \vdots & 5 \\ 3 & 4 & 2 & \vdots & 11 \\ 1 & 2 & -2 & \vdots & -3 \end{bmatrix}$$

We apply the elementary row operations to the above matrix to reduce it to the equivalent reduced (row) echelon form, that is,

$$\begin{bmatrix} 2 & 5 & -1 & \vdots & 5 \\ 3 & 4 & 2 & \vdots & 11 \\ 1 & 2 & -2 & \vdots & -3 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 3 & 4 & 2 & \vdots & 11 \\ 2 & 5 & -1 & \vdots & 5 \end{bmatrix} \text{ BY } R_1 \leftrightarrow R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 0 & -2 & 8 & \vdots & 20 \\ 2 & 5 & -1 & \vdots & 5 \end{bmatrix} \text{BY } R_2 + (-3)R_1 \to R_2' \; \underset{\sim}{R} \begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 0 & -2 & 8 & \vdots & 20 \\ 0 & 1 & 3 & \vdots & 11 \end{bmatrix} \text{BY } R_3 + (-2)R_1 \to R_3'$$

BY $-\frac{1}{2}R_2 \to R_2'$, we get

$$\begin{bmatrix} 1 & 2 & -2 & \vdots & -3 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 1 & 3 & \vdots & 11 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 6 & \vdots & 17 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 0 & 7 & \vdots & 21 \end{bmatrix} \begin{array}{l} \text{By } R_1 + (-2)R_2 \to R'_1 \\ \text{and } R_3 + (-1), R_2 \to R'_3 \end{array}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 6 & \vdots & 17 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \text{BY } \frac{1}{7}R3 \to R'3 \; \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \begin{array}{l} By\ R_1 + (-6)R_3 \to R'_1 \\ \text{and } R_2 + 4R_3, R'_2 \end{array}$$

Thus the solution is $x = -1$, $y = 2$ and $z = 3$.

The augmented matrix for the system (2) is

$$\begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 2 & -1 & 7 & \vdots & 11 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix}$$

Adding $(-2)R_1$ to R_2 and $(-3)R_1$ to R_3, we get

$$\begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 2 & -1 & 7 & \vdots & 11 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix} \qquad \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 0 & -3 & 3 & \vdots & 9 \\ 0 & 2 & -2 & \vdots & -6 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 2 & -2 & \vdots & -6 \end{bmatrix} \text{By } -\frac{1}{3}R_2 \to R_2' \; \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 3 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \begin{array}{l} \text{BY } R_1 + (-1)R_2 \to R_1' \\ \text{and } R_3 + (-2)R_2 \to R_3' \end{array}$$

The system (2) is reduced to equivalent system

$$x + 3z = 4$$
$$y - z = -3$$
$$0z = 0$$

The equation $0z = 0$ is satisfied by any value of z.

From the first and second equations, we get

$$x = -3z + 4 \quad ....(a)$$

and $\quad y = z - 3 \quad ....(b)$

As z is arbitrary, so we can find infinitely many values of x and y from equation (a) and (b) or the system (2), is satisfied by $x = 4 - 3t$, $y = t - 3$ and $z = t$ for any real value of t.

Thus the system (2) has infinitely many solutions and it is consistent.

The augmented matrix of the system (3) is $\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 2 & -6 & 5 & \vdots & 7 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix}$

Adding $(-2)R_1$, to R_2 and $(-3)R_1$, to R_3 we have

$$\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 2 & -6 & 5 & \vdots & 7 \\ 3 & 5 & 4 & \vdots & -3 \end{bmatrix} \quad \underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 0 & -4 & 1 & \vdots & 5 \\ 0 & 8 & -2 & \vdots & -6 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & \vdots & 1 \\ 0 & 1 & -\frac{1}{4} & \vdots & -\frac{5}{4} \\ 0 & 8 & -2 & \vdots & -6 \end{bmatrix} \text{By } -\frac{1}{4}R_2 \to R_2' \quad \underset{\sim}{R} \begin{bmatrix} 1 & 0 & \frac{7}{4} & \vdots & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & \vdots & -\frac{5}{4} \\ 0 & 0 & 0 & \vdots & 4 \end{bmatrix} \begin{matrix} \text{By } R_1 + 1.R_2 \to R_1' \\ \text{and } R_3 + (-8)R_2 \to R_3' \end{matrix}$$

Thus the system (3) is reduced to the equivalent system

$$x + \frac{7}{4}z = \frac{1}{4}$$
$$y - \frac{1}{4}z = \frac{5}{4}$$
$$0z = 4$$

The third equation $0z = 4$ has no solution, so the system as a whole has no solution. Thus the system is inconsistent.

We see that in the case of the system (1), the (row) rank of the augmented matrix and the coefficient matrix of the system is the same, that is, 3 which is equal to the number of the variables in the system (1).

Thus a linear system is consistent and has a unique solution if the
(row) rank of the coefficient matrix is the same as that of the augmented matrix of the system.

In the case of the system (2), the (row) rank of the coefficient matrix is the same as that of the augmented matrix of the system but it is 2 which is less than the number of variables in the system (2).

Thus a system is consistent and has infinitely many solutions if the (row) ranks of the coefficient matrix and the augmented matrix of the system are equal but the rank is less than the number of variables in the system.

In the case of the system (3), we see that the (row) rank of the coefficient matrix is not equal to the (row) rank of the augmented matrix of the system.

Thus we conclude that a system is inconsistent if the (row) ranks of the coefficient matrix and the augmented matrix of the system are different.

3.11.1 Homogeneous Linear Equations

Each equation of the system of following linear equations:

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 & ....(i) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 & ....(ii) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0 & ....(iii) \end{matrix}\right\}$$

is always satisfied by $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, so such a system is always consistent. The solution (0, 0, 0) of the above homogeneous equations (i), (ii), and (iii) is called the trivial solution. Any other solution of equations (i), (ii) and (iii) other than the trivial solution is called a **non-trivial**
solution. The above system can be written as

$$AX = O \text{, where } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $|A| \neq 0$, then A is non-singular and A^{-1} exists, that is,

$$A^{-1}(AX) = A^{-1}O = O$$

or $(A^{-1}A)X = O \Rightarrow X = O$, i.e., $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

In this case the system of homogeneous equations possesses only the trivial solution.

Now we consider the case when the system has a non-trivial solution.

Multiplying the equations (i), (ii) and (iii) by A_{11}, A_{21} and A_{31} respectively and adding the resulting equations (where A_{11}, A_{21} and A_{31} are cofactors of the corresponding elements of A), we have

$(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31})x_1 + (a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31})x_2 + (a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31})x_3 = 0$, that is, $|A|x_1 = 0$. Similarly, we can get $|A|x_2 = 0$ and $|A|x_3 = 0$

For a non-trivial solution, at least one of x_1, x_2 and x_3 is different from zero. Let $x_1 \neq 0$, then from $|A|x_1 = 0$, we have $|A| = 0$.

For example, the system

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 0 \quad (\text{I}) \\ x_1 - x_2 + 3x_3 &= 0 \quad (\text{II}) \\ x_1 + 3x_2 - x_3 &= 0 \quad (\text{III}) \end{aligned}\right\}$$

has a non-trivial solution because

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = 0$$

Solving the first two equations of the system, we have

$2x_1 + 4x_3 = 0$ (adding (I) and (II))

$\Rightarrow \quad x_1 = -2x_3$

and $2x_2 - 2x_3 = 0$ (subtracting (II) from (I))

$\Rightarrow \quad x_2 = x_3$

Putting $x_1 = -2x_3$ and $x_2 = x_3$ in (III), we see that $(-2x_3) + 3(x_3) - x_3 = 0$, which shows that the equation (I), (II) and (III) are satisfied by

$x_1 = -2t$, $x_2 = t$ and $x_3 = t$ for any real value of t.

Thus the system consisting of (I), (II) and (III) has infinitely many solutions. But the system

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + 3x_3 &= 0 \\ x_1 + 3x_2 - 2x_3 &= 0 \end{aligned}\right\}$$ has only the trivial solution,

because in this case

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

Solving the first two equations of the above system, we get $x_1 = -2x_3$ and $x_2 = x_3$. Putting $x_1 = -2x_3$ and $x_2 = x_3$ in the expression.

$x_1 + 3x_2 - 2x_3$, we have $-2x_3 + 3(x_3) - 2x_3 = -x_3$, that is,

the third equation is not satisfied by putting $x_1 = -2x_3$ and $x_2 = x_3$ but it is satisfied only if $x_3 = 0$. Thus the above system has only the trivial solution.

3.11.2 Non-Homogeneous Linear Equations

Now we will solve the systems of non-homogeneous linear equations with help of the following methods.

i) Using matrices, that is, $AX = B \Rightarrow X = A^{-1}B$.

ii) Using echelon and reduced echelon forms

iii) Using Cramer's rule.

Example 1: Use matrices to solve the system $\left.\begin{aligned} x_1 - 2x_2 + x_3 &= -4 \\ 2x_1 - 3x_2 + 2x_3 &= 6 \\ 2x_1 + 2x_2 + x_3 &= 5 \end{aligned}\right\}$

Solution: The matrix form of the given system is

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix}$$

or $\quad AX = B \qquad$ (i)

where $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix}$

As $\quad |A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{vmatrix}$ By $R_2 + (-2)R_1 \rightarrow R_2'$

$$= (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1-2) = -1 \text{, that is,}$$

$|A| \neq 0$, so the inverse of A exists and (i) can be written as

$X = A^{-1}B \qquad$ (ii)

Now we find adj A.

Since $\quad [A_{ij}]_{3\times 3} = \begin{bmatrix} -7 & 2 & 10 \\ 4 & -1 & -6 \\ -1 & 0 & 1 \end{bmatrix}, \left(\because A_{11} = -7, A_{12} = 2, A_{13} = 10, A_{21} = 4, A_{22} = -1, A_{23} = -6, A_{31} = 1, A_{32} = 0, A_{33} = 1 \right)$

So $\quad$ adj $A = \begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix}$

and $\quad A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{-1}\begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}$

Thus $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -28+24+5 \\ 8-6+0 \\ 40-36-5 \end{bmatrix}$, i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence $x_1 = 1$, $x_2 = 2$ and $x_3 = -1$.

Example 2: Solve the system;

$$\left.\begin{aligned} x_1 + 3x_2 + 2x_3 &= 3 \\ 4x_1 + 5x_2 - 3x_3 &= -3 \\ 3x_1 - 2x_2 + 17x_3 &= 42 \end{aligned}\right\},$$

by reducing its augmented matrix to the echelon form and the reduced echelon form.

Solution: The augmented matrix of the given system is

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 4 & 5 & -3 & \vdots & -3 \\ 3 & -2 & 17 & \vdots & 42 \end{bmatrix}$$

We reduce the above matrix by applying elementary row operations, that is,

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 4 & 5 & -3 & \vdots & -3 \\ 3 & -2 & 17 & \vdots & 42 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & -7 & -11 & \vdots & -15 \\ 0 & -11 & 11 & \vdots & 33 \end{bmatrix} \begin{matrix} \text{By } R_2 + (-4)R_1 \rightarrow R_2' \\ \text{and } R_3 + (-3)R_1 \rightarrow R_3' \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & -11 & 11 & \vdots & 33 \\ 0 & -7 & -11 & \vdots & -15 \end{bmatrix} \text{By } R_2 \leftrightarrow R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & -7 & -11 & \vdots & -15 \end{bmatrix} \text{By } \left(-\frac{1}{11}\right)R_2 \rightarrow R_2' \; \underset{\sim}{R} \begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & -18 & \vdots & -36 \end{bmatrix} \text{By } R_3 + 7R_2 \rightarrow R_3'$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \text{By} \left(-\frac{1}{18}\right)R_3 \rightarrow R_3'$$

The equivalent system in the (row) echelon form is

$$\left.\begin{array}{r} x_1 + 3x_2 + 2x_3 = 3 \\ x_2 - x_3 = -3 \\ x_3 = 2 \end{array}\right\}$$

Substituting $x_3 = 2$ in the second equation gives: $x_2 - 2 = -3 \Rightarrow x_2 = -1$
Putting $x_2 = -1$ and $x_3 = 2$ in the first equation, we have

$$x_1 + 3(-1) + 2(2) = 3 \Rightarrow x_1 = 3 + 3 - 4 = 2.$$

Thus the solution is $x_1 = 2$, $x_2 = -1$ and $x_3 = 2$

Now we reduce the matrix $\begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & -1 & \vdots & 2 \end{bmatrix}$ to reduced (row) echelon form, i.e.,

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 0 & 5 & \vdots & 12 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \quad \text{By } R_1 + (-3)R_2 \to R_1'$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \quad \begin{array}{l} \text{By } R_1 + (-5)R_3 \to R_1' \\ \text{and } R_2 + 1.R_3 \to R_2' \end{array}$$

The equivalent system in the reduced (row) echelon form is

$x_1 = 2$
$x_2 = -1$
$x_3 = 2$

which is the solution o f the given system.

3.12 Cramer's Rule

Consider the system of equations,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}\right\} \qquad (1)$$

These are three linear equations in three variables x_1, x_2, x_3 with coefficients and constant terms in the real field R. We write the above system of equations in matrix form as: $AX = B$ (2)

where $A = [a_{ij}]_{3\times 3}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

We know that
the matrix equation (2) can be written as: $X = A^{-1}B$ (if A^{-1} exists)

Note: $A^{-1}(AX) \neq BA^{-1}$

We have already proved that $A^{-1} = \dfrac{1}{|A|}\text{adj } A$ and

$$adj\ A = [A'_{ij}]_{3\times 3} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \qquad (\because A'_{ij} = A_{ji})$$

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \dfrac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \dfrac{1}{|A|}\begin{bmatrix} A_{11}b_1 + A_{21}b_2 + A_{31}b_3 \\ A_{12}b_1 + A_{22}b_2 + A_{32}b_3 \\ A_{13}b_1 + A_{23}b_2 + A_{33}b_3 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \dfrac{A_{11}b_1 + A_{21}b_2 + A_{31}b_3}{|A|} \\ \dfrac{A_{12}b_1 + A_{22}b_2 + A_{32}b_3}{|A|} \\ \dfrac{A_{13}b_1 + A_{23}b_2 + A_{33}b_3}{|A|} \end{bmatrix}$$

Hence $x_1 = \dfrac{b_1A_{11} + b_2A_{21} + b_3A_{31}}{|A|} = \dfrac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}$ (i)

$$x_2 = \frac{b_1A_{12} + b_2A_{22} + b_3A_{32}}{|A|} = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|} \qquad \text{(ii)}$$

$$x_3 = \frac{b_1A_{13} + b_2A_{23} + b_3A_{33}}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|} \qquad \text{(iii)}$$

The method of solving the system with the help of results (i), (ii) and (iii) is often referred to as **Cramer's Rule.**

Example 3: Use Crammer's rule to solve the system. $\left.\begin{aligned} 3x_1 + x_2 - x_3 &= -4 \\ x_1 + x_2 - 2x_3 &= -4 \\ -x_1 + 2x_2 - x_3 &= 1 \end{aligned}\right\}$

Solution: Here $|A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3(-1+4) - 1.(-1-2) - 1.(2+1)$

$= 9 + 3 - 3 = 9$

So $x_1 = \frac{\begin{vmatrix} -4 & 1 & -1 \\ -4 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix}}{9} = \frac{-4(-1+4) - 1(4+2) - 1(-8-1)}{9}$

$= \frac{-12 - 6 + 9}{9} = \frac{-9}{9} = 1$

$x_2 = \frac{\begin{vmatrix} 3 & -4 & -1 \\ 1 & -4 & -2 \\ -1 & 1 & -1 \end{vmatrix}}{9} = \frac{3(4+2) + 4(-1-2) - 1(1-4)}{9}$

$= \frac{18 - 12 + 3}{9} = \frac{9}{9} = 1$

$x_3 = \frac{\begin{vmatrix} 3 & 1 & -4 \\ 1 & 1 & -4 \\ -1 & 2 & 1 \end{vmatrix}}{9} = \frac{3(1+8) - 1(1-4) - (2+1)}{9}$

$= \frac{27 + 3 - 12}{9} = \frac{18}{9} = 2$

Hence $x_1 = -1$, $x_2 = 1$, $x_3 = 2$

Exercise 3.5

1. Solve the following systems of linear equations by Cramer's rule.

i) $\left.\begin{aligned} 2x + 2y + z &= 3 \\ 3x - 2y - 2z &= 1 \\ 5x + y - 3z &= 2 \end{aligned}\right\}$ ii) $\left.\begin{aligned} 2x_1 - x_2 + x_3 &= 5 \\ 4x_1 + 2x_2 + 3x_3 &= 8 \\ 3x_1 - 4x_2 - x_3 &= 3 \end{aligned}\right\}$ iii) $\left.\begin{aligned} 2x_1 - x_2 + x_3 &= 8 \\ x_1 + 2x_2 + 2x_3 &= 6 \\ x_1 - 2x_2 - x_3 &= 1 \end{aligned}\right\}$

2. Use matrices to solve the following systems:

i) $\left.\begin{aligned} x - 2y + z &= -1 \\ 3x + y - 2z &= 4 \\ y - z &= 1 \end{aligned}\right\}$ ii) $\left.\begin{aligned} 2x_1 + x_2 + 3x_3 &= 3 \\ x_1 + x_2 - 2x_3 &= 0 \\ -3x_1 - x_2 + 2x_3 &= -4 \end{aligned}\right\}$ iii) $\left.\begin{aligned} x + y &= 2 \\ 2x - z &= 1 \\ 2y - 3z &= -1 \end{aligned}\right\}$

3. Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms.

i) $\left.\begin{aligned} x_1 - 2x_2 - 2x_3 &= 1 \\ 2x_1 + 3x_2 + x_3 &= 1 \\ 5x_1 - 4x_2 - 3x_3 &= 1 \end{aligned}\right\}$ ii) $\left.\begin{aligned} x + 2y + z &= 2 \\ 2x + y + 2z &= -1 \\ 2x + 3y - z &= 9 \end{aligned}\right\}$ iii) $\left.\begin{aligned} x_1 + 4x_2 + 2x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= 9 \\ 3x_1 + 2x_2 - 2x_3 &= 12 \end{aligned}\right\}$

4. Solve the following systems of homogeneous linear equations.

i) $\left.\begin{aligned} x + 2y - 2z &= 0 \\ 2x + y + 5z &= 0 \\ 5x + 4y + 8z &= 0 \end{aligned}\right\}$ ii) $\left.\begin{aligned} x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ 3x_1 + 2x_2 - 4x_3 &= 0 \end{aligned}\right\}$ iii) $\left.\begin{aligned} x_1 - 2x_2 - x_3 &= 0 \\ x_1 + x_2 + 5x_3 &= 0 \\ 2x_1 - x_2 + 4x_3 &= 0 \end{aligned}\right\}$

5. Find the value of λ for which the following systems have non-trivial solutions. Also solve the system for the value of λ.

i) $$\left.\begin{array}{r} x+y+z=0 \\ 2x+y-\lambda z=0 \\ x+2y-2z=0 \end{array}\right\}$$

ii) $$\left.\begin{array}{r} x_1+4x_2+\lambda x_3=0 \\ 2x_1+x_2-3x_3=0 \\ 3x_1+\lambda x_2-4x_3=0 \end{array}\right\}$$

6. Find the value of λ for which the following system does not possessa unique solution. Also solve the system for the value of λ.

$$\left.\begin{array}{r} x_1+4x_2+\lambda x_3=2 \\ 2x_1+x_2-2x_3=11 \\ 3x_1+2x_2-2x_3=16 \end{array}\right\}$$

Quadratic Equations

Animation 4.1: Completing the square
Source & Credit: 1ucasvb

4.1 Introduction

A *quadratic equation* in x is an equation that can be written in the form $ax^2 + bx + c = 0$; where a, b and c are real numbers and
$a \neq 0$.

Another name for a quadratic equation in x is **2nd Degree Polynomial** in x.

The following equations are the quadratic equations:

i)	$x^2 - 7x + 10 = 0$;	$a = 1$,	$b = -7$,	$c = 10$
ii)	$6x^2 + x - 15 = 0$;	$a = 6$,	$b = 1$,	$c = -15$
iii)	$4x^2 + 5x + 3 = 0$;	$a = 4$,	$b = 5$,	$c = 3$
iv)	$3x^2 - x = 0$;	$a = 3$,	$b = -1$,	$c = 0$
v)	$x^2 = 4$;	$a = 1$,	$b = 0$,	$c = -4$

4.1.1 Solution of Quadratic Equations

There are three basic techniques for solving a quadratic equation:

i) by factorization.
ii) by completing squares, extracting square roots.
iii) by applying the quadratic formula.

By Factorization: It involves factoring the polynomial $ax^2 + bx + c$.

It makes use of the fact that if $ab = 0$, then $a = 0$ or $b = 0$.

For example, if $(x - 2)(x - 4) = 0$, then either $x - 2 = 0$ or $x - 4 = 0$.

Example 1: Solve the equation $x^2 - 7x + 10 = 0$ by factorization.

Solution: $x^2 - 7x + 10 = 0$

$\Rightarrow$ $(x - 2)(x - 5) = 0$

$\therefore$ either $x - 2 = 0$ $\Rightarrow x = 2$

or $x - 5 = 0$ $\Rightarrow x = 5$

$\therefore$ the given equation has two solutions: 2 and 5

$\therefore$ solution set = $\{2, 5\}$

Note: The solutions of an equation are also called its roots.

$\therefore$ 2 and 5 are roots of $x^2 - 7x + 10 = 0$

By Completing Squares, then Extracting Square Roots:

Sometimes, the quadratic polynomials are not easily factorable.

For example, consider $x^2 + 4x - 437 = 0$.

It is difficult to make factors of $x^2 + 4x - 437$. In such a case the factorization and hence the solution of quadratic equation can be found by the method of completing the square and extracting square roots.

Example 2: Solve the equation $x^2 + 4x - 437 = 0$ by completing the squares.

Solution : $x^2 + 4x - 437 = 0$

$$\Rightarrow \quad x^2 + 2\left(\frac{4}{2}\right)x = 437$$

Add $\left(\frac{4}{2}\right)^2 = (2)^2$ to both sides

$x^2 + 4x + (2)^2 = 437 + (2)^2$

$\Rightarrow$ $(x + 2)^2 = 441$

$\Rightarrow$ $x + 2 = \pm\sqrt{441} = \pm 21$

$\Rightarrow$ $x = \pm 21 - 2$

$\therefore$ $x = 19$ or $x = -23$

Hence solution set = $\{-23, 19\}$.

By Applying the Quadratic Formula:

Again there are some quadratic polynomials which are not factorable at all using integral coefficients. In such a case we can always find the solution of a quadratic equation $ax^2+bx+c = 0$ by applying a formula known as quadratic formula. This formula is applicable for every quadratic equation.

Derivation of the Quadratic Formula

Standard form of quadratic equation is

$ax^2 + bx + c = 0, a \neq 0$

Step 1. Divide the equation by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Step 2. Take constant term to the R.H.S.

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Step 3. To complete the square on the L.H.S. add $\left(\frac{b}{2a}\right)^2$ to both sides.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Hence the solution of the quadratic equation $ax^2+bx+c=0$ is given by

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

which is called **Quadratic Formula.**

Example 3: Solve the equation $6x^2+x-15=0$ by using the quadratic formula.

Solution: Comparing the given equation with $ax^2+bx+c=0$, we get,

$a=6$, $b=1$, $c=-15$

$\therefore$ The solution is given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-1\pm\sqrt{1^2-4(6)(-15)}}{2(6)}$$

$$=\frac{-1\pm\sqrt{361}}{12}=\frac{-1\pm 19}{12}$$

i.e., $x=\frac{-1+19}{12}$ or $x=\frac{-1-19}{12}$

$x=\frac{3}{2}$ or Hence soulation set $=\left\{\frac{3}{2},\frac{-5}{3}\right\}$

Example 4: Solve the $8x^2-14x-15=0$ by using the quadratic formula.

Solution: Comparing the given equation with $ax^2+bx+c=0$, we get,

$a=8$, $b=-14$, $c=-15$

By the quadratic formula, we have

$$\therefore \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x=\frac{-(-14)\pm\sqrt{(-14)^2-4(8)(-15)}}{2(8)}$$

$$=\frac{14\pm\sqrt{676}}{16}=\frac{14\pm 26}{16}$$

$\therefore$ either $x=\frac{14+26}{16}\Rightarrow x=\frac{5}{2}$

or $\Rightarrow$

Hence solution set $=\left\{\frac{5}{2},\ -\frac{3}{4}\right\}$

Exercise 4.1

Solve the following equations by factorization:

1. $3x^2+4x+1=0$ **2.** $x^2+7x+12=0$

3. $9x^2-12x-5=0$ **4.** $x^2-x=2$

5. $x(x + 7) = (2x - 1)(x + 4)$

6. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{5}{2};\ x\neq -1,0$

7. $\frac{1}{x+1}+\frac{2}{x+2}=\frac{7}{x+5};\ x\neq -1,-2,-5$

8. $\frac{a}{ax-1}+\frac{b}{bx-1}=a+b;\ x\neq \frac{1}{a},\frac{1}{b}$

Solve the following equations by completing the square:

9. $x^2 - 2x - 899 = 0$
10. $x^2 + 4x - 1085 = 0$
11. $x^2 + 6x - 567 = 0$
12. $x^2 - 3x - 648 = 0$
13. $x^2 - x - 1806 = 0$
14. $2x^2 + 12x - 110 = 0$

Find roots of the following equations by using quadratic formula:

15. $5x^2 - 13x + 6 = 0$
16. $4x^2 + 7x - 1 = 0$
17. $15x^2 + 2ax - a^2 = 0$
18. $16x^2 + 8x + 1 = 0$
19. $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$
20. $(a + b)x^2 + (a + 2b + c)x + b + c = 0$

4.2 Solution of Equations Reducible to the Quadratic Equation

There are certain types of equations, which do not look to be of degree 2, but they can be reduced to the quadratic form. We shall discuss the solutions of such five types of the equations one by one.

Type I: The equations of the form: $ax^{2n} + bx^n + c = 0;\ a \neq 0$

Put $x^n = y$ and get the given equation reduced to quadratic equation in y.

Example 1: Solve the equation: $x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0.$

Solution This given equation can be written as $(x^{\frac{1}{4}})^2 - x^{\frac{1}{4}} - 6 = 0$

Let $x^{\frac{1}{4}} = y$

$\therefore$ The given equation becomes

$y^2 - y - 6 = 0$

$\Rightarrow (y - 3)(y + 2) = 0$

$\Rightarrow y = 3,$ or $y = -2$

$\therefore x^{\frac{1}{4}} = 3$ $\quad x^{\frac{1}{4}} = -2$

$\Rightarrow x = (3)^4$ $\quad \Rightarrow x = (-2)^4$

$\Rightarrow x = 81$ $\quad \Rightarrow x = 16$

Hence solution set is {16, 81}.

Type II: The equation of the form: $(x + a)(x + b)(x + c)(x + d) = k$
where $a + b = c + d$

Example 2: Solve $(x - 7)(x - 3)(x + 1)(x + 5) - 1680 = 0$

Solution: $(x - 7)(x - 3)(x + 1)(x + 5) - 1680 = 0$

$\Rightarrow [(x - 7)(x + 5)][(x - 3)(x + 1)] - 1680 = 0$ (by grouping)

$\Rightarrow (x^2 - 2x - 35)(x^2 - 2x - 3) - 1680 = 0$

Putting $x^2 - 2x = y$, the above equation becomes

$(y - 35)(y - 3) - 1680 = 0$

$\Rightarrow y^2 - 38y + 105 - 1680 = 0$

$\Rightarrow y^2 - 38y - 1575 = 0$

$\therefore y = \frac{38 \pm \sqrt{1444 + 6300}}{2} = \frac{38 \pm \sqrt{7744}}{2}$ (by quadratic formula)

$= \frac{38 \pm 88}{2}$

$\Rightarrow$ $y = 63$ or $y = -25.$

$\Rightarrow$ $x^2 - 2x = 63$ | $\Rightarrow x^2 - 2x = -25$

$\Rightarrow$ $x^2 - 2x - 63 = 0$ | $\Rightarrow x^2 - 2x + 25 = 0$

$\Rightarrow$ $(x+7)(x-9) = 0$ | $\Rightarrow x = \dfrac{2 \pm \sqrt{4-100}}{2}$

$\Rightarrow$ $x = -7$ or $x = 9$ | $= \dfrac{2 \pm \sqrt{-96}}{2}$

$= \dfrac{2 \pm 4\sqrt{6}\,i}{2} = 1 \pm 2\sqrt{6}\,i$

$\Rightarrow$ or

Hence Solution set = $\{-7, 9, 1+2\sqrt{6}\,i, 1-2\sqrt{6}\,i\}$

Type III: Exponential Equations: Equations, in which the variable occurs in exponent, are called **exponential equations**. The method of solving such equations is explained by the following examples.

Example 3: Solve the equation: $2^{2x} - 3.2^{x+2} + 32 = 0$

Solution: $2^{2x} - 3.2^{x+2} + 32 = 0$

$\Rightarrow$ $2^{2x} - 3.2^2 . 2^x + 32 = 0$

$\Rightarrow$ $2^{2x} - 12.2^x + 32 = 0$

$\Rightarrow$ $y^2 - 12y + 32 = 0$ (Putting $2^x = y$)

$\Rightarrow$ $(y-8)(y-4) = 0$

$\Rightarrow$ $y = 8$ or $y = 4$

$\Rightarrow$ $2^x = 8$ $\Rightarrow$ $2^x = 4$

$\Rightarrow$ $2^x = 2^3$ $\Rightarrow$ $2^x = 2^2$

$\Rightarrow$ $x = 3$ $\Rightarrow$ $x = 2$

Hence solution set = {2, 3}.

Example 4: Solve the equation: $4^{1+x} + 4^{1-x} = 10$

Solution: Given that

$4^{1+x} + 4^{1-x} = 10$

$\Rightarrow$ $4.4^x + 4.4^{-x} = 10$

Let $4^x = y \Rightarrow 4^{-x} = (4^x)^{-1} = y^{-1} = \dfrac{1}{y}$

$\therefore$ The given equation becomes

$4y + \dfrac{4}{y} - 10 = 0$

$\Rightarrow$ $4y^2 - 10y + 4 = 0$

$\Rightarrow$ $2y^2 - 5y + 2 = 0$

$\therefore$ $y = \dfrac{5 \pm \sqrt{25 - 4(2)(2)}}{2(2)} = \dfrac{5 \pm \sqrt{9}}{4} = \dfrac{5 \pm 3}{4}$

$\Rightarrow$ $y = 2$ or $y = \dfrac{1}{2}$

$\therefore$ $4^x = 2$ | $\therefore$ $4^x = \dfrac{1}{2}$

$\Rightarrow$ $2^{2x} = 2^1$

$\Rightarrow$ $2x = 1$ | $\Rightarrow$ $2^{2x} = 2^{-1}$

| $\Rightarrow$ $2x = -1$

$\Rightarrow$ $x = \dfrac{1}{2}$ | $\Rightarrow$

Hence Solution set = $\left\{\dfrac{1}{2}, -\dfrac{1}{2}\right\}$.

Type IV: Reciprocal Equations: An equation, which remains unchanged when x is replaced by is called a reciprocal equation. In such an equation the coefficients of the terms equidistant from the beginning and end are equal in magnitude. The method of solving such equations is explained through the following example:

Example 5: Solve the equation

$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$;

Solution: Given that:

$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$

$\Rightarrow \quad x^2 - 3x + 4 - \frac{3}{x} + \frac{1}{x^2} = 0$ (Dividing by x^2)

$\Rightarrow \quad \left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) + 4 = 0$ (1)

Let $\Rightarrow$

So, the equation (1) reduces to

$y^2 - 2 - 3y + 4 = 0$

$\Rightarrow \quad y^2 - 3y + 2 = 0$

$\Rightarrow \quad (y - 2)(y - 1) = 0$

$\Rightarrow \quad y = 2$ or $y = 1$

$\Rightarrow \quad x + \frac{1}{x} = 2 \qquad \Rightarrow \quad x + \frac{1}{x} = 1$

$\Rightarrow \quad x^2 - 2x + 1 = 0 \qquad \Rightarrow \quad x^2 - x + 1 = 0$

$\Rightarrow \quad (x - 1)^2 = 0 \qquad \Rightarrow$

$\Rightarrow \quad (x - 1)(x - 1) = 0$

$\Rightarrow \quad x = 1, 1 \qquad \Rightarrow$

Hence Solution set

Exercise 4.2

Solve the following equations:

1. $x^4 - 6x^2 + 8 = 0$ **2.** $x^{-2} - 10 = 3x^{-1}$

3. $x^{-6} - 9x^3 + 8 = 0$ **4.** $8x^6 - 19x^3 - 27 = 0$

5. $x^{\frac{2}{5}} + 8 = 6x^{\frac{1}{5}}$ **6.** $(x+1)(x+2)(x+3)(x+4) = 24$

7. $(x-1)(x+5)(x+8)(x+2) - 880 = 0$

8. $(x-5)(x-7)(x+6)(x+4) - 504 = 0$

9. $(x-1)(x-2)(x-8)(x+5) + 360 = 0$

10. $(x+1)(2x+3)(2x+5)(x+3) = 945$

Hint: $(x+1)(2x+5)(2x+3)(x+3) = 945$

11. $(2x-7)(x^2-9)(2x+5) - 91 = 0$

12. $(x^2+6x+8)(x^2+14x+48) = 105$

13. $(x^2+6x-27)(x^2-2x-35) = 385$

14. $4 \cdot 2^{2x+1} - 9.2^x + 1 = 0$

15. $2^x + 2^{-x+6} - 20 = 0$

16. $4^x - 3.2^{x+3} + 128 = 0$

17. $3^{2x-1} - 12.3^x + 81 = 0$ **18.**

19. $x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0$ **20.** $\left(x - \frac{1}{x}\right)^2 + 3\left(x + \frac{1}{x}\right) = 0$

21. $2x^4 - 3x^3 - x^2 - 3x + 2 = 0$ **22.** $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

23. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$ **24.** $x^4 - 6x^2 + 10 - \frac{6}{x^2} + \frac{1}{x^4} = 0$

Type V: Radical Equations: Equations involving **radical expressions** of the variable are called radical equations. To solve a radical equation, we first obtain an equation free from radicals. Every solution of radical equation is also a solution of the radical-free equation but the new equation have solutions that are not solutions of the original radical equation.

Such extra solutions (roots) are called **extraneous roots**. The method of the solution of different types of radical equations is illustrated by means of the followings examples:

i) The Equations of the form: $l(ax^2+bx)+m\sqrt{ax^2+bx+c}=0$

Example 1: Solve the equation

$$3x^2+15x-2\sqrt{x^2+5x+1}=2$$

Solution : Let $\sqrt{x^2+5x+1}=y$

$\Rightarrow$ $x^2+5x+1=y^2$

$\Rightarrow$ $x^2+5x=y^2-1$

$\Rightarrow$ $3x^2+15x=3y^2-3$

$\therefore$ The given equation becomes $3y^2-3-2y=2$

$\Rightarrow$ $3y^2-2y-5=0$

$\Rightarrow$ $(3y-5)(y+1)=0$

$\Rightarrow$ $y=\frac{5}{3}$ or $y=-1$

$\Rightarrow \sqrt{x^2+5x+1}=\frac{5}{3}$	$\Rightarrow \sqrt{x^2+5x+1}=-1$
$\Rightarrow x^2+5x+1=\frac{25}{9}$	$\Rightarrow x^2+5x+1=1$
$\Rightarrow 9x^2+45x+9=25$	$\Rightarrow x^2+5x=0$
$\Rightarrow 9x^2+45x-16=0$	$\Rightarrow x(x+5)=0$
$\Rightarrow (3x+16)(3x-1)=0$	$\therefore x=0$ or $x=-5$

$\therefore$ $x=\frac{1}{3}$ or $x=-\frac{16}{3}$

On checking, it is found that 0 and – 5 do not satisfy the given equation. Therefore 0 and –5 being extraneous roots cannot be included in solution set.

Hence solution set

ii) The Equation of the form : $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$

Example 2: Solve the equation:

Solution: $\sqrt{x+8}+\sqrt{x+3}=\sqrt{12x+13}$

Squaring both sides, we get

$$x+8+x+3+2\sqrt{x+8}\sqrt{x+3}=12x+13$$

$\Rightarrow$ $2\sqrt{x+8}\sqrt{x+3}=10x+2$

$\Rightarrow$ $\sqrt{(x+8)(x+3)}=5x+1$

Squaring again, we have

$x^2+11x+24=25x^2+10x+1$

$\Rightarrow$ $24x^2-x-23=0$

$\Rightarrow$ $(24x+23)(x-1)=0$

$\Rightarrow$ $x=-\frac{23}{24}$ or $x=1$

On checking we find that $-\frac{23}{24}$ is an extraneous root. Hence solution set = {1}.

iii) The Equations of the form:

$$\sqrt{ax^2+bx+c}+\sqrt{px^2+qx+r}=\sqrt{lx^2+mx+n}$$

where ax^2+bx+c, px^2+qx+r and lx^2+mx+n have a common factor.

Example3: Solve the equation: $\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$

Solution: Consider that:

$x^2+4x-21=(x+7)(x-3)$

$x^2-x-6=(x+2)(x-3)$

$6x^2-5x-39=(6x+13)(x-3)$

$\therefore$ The given equation can be written as

$$\sqrt{(x+7)(x-3)}+\sqrt{(x+2)(x-3)}=\sqrt{(6x+13)(x-3)}$$

$\Rightarrow$ $\sqrt{x-3}\left[\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}\right]=0$

$\therefore$ Either $\sqrt{x-3}=0$ or $\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}=0$

$\sqrt{x-3}=0 \Rightarrow x-3=0 \Rightarrow x=3$

Now solve the equation $\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}=0$

$\Rightarrow \quad \sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

$\Rightarrow \quad x+7+x+2+2\sqrt{(x+7)(x+2)}=6x+13$ (Squaring both sides)

$\Rightarrow \quad 2\sqrt{(x+7)(x+2)}=4x+4$

$\Rightarrow \quad \sqrt{x^2+9x+14}=2x+2$

$\Rightarrow \quad x^2+9x+14 = 4x^2+8x+4$ (Squaring both sides again)

$\Rightarrow \quad 3x^2-x-10=0$

$\Rightarrow \quad (3x+5)(x-2)=0$

$\Rightarrow \quad x=-\frac{5}{3},2$

Thus possible roots are 3, 2, $-\frac{5}{3}$.

On verification, it is found that $-\frac{5}{3}$ is an extraneous root. Hence solution set = {2, 3}

iv) The Equations of the form: $\sqrt{ax^2+bx+c}+\sqrt{px^2+qx+r}=mx+n$
where, $(mx+n)$ is a factor of $(ax^2+bx+c)-(px^2+qx+r)$

Example 4: Solve the equation: $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5$

Solution: Let $\sqrt{3x^2-7x-30}=a$ and $\sqrt{2x^2-7x-5}=b$

Now $a^2-b^2=(3x^2-7x-30)-(2x^2-7x-5)$

$a^2-b^2=x^2-25$ (i)

The given equation can be written as:

$a-b=x-5$ (ii)

$\frac{(a+b)(a-b)}{a-b}=\frac{(x+5)(x-5)}{x-5}$ [From (i) and (ii)]

$\Rightarrow \quad a+b=x+5$ (iii)

$2a=2x$ [From (ii) and (iii)]

$\Rightarrow \quad a=x$

$\therefore \quad \sqrt{3x^2-7x-30}=x$

$\Rightarrow \quad 3x^2-7x-30=x^2$

$\Rightarrow \quad 2x^2-7x-30=0$

$\Rightarrow \quad (2x+5)(x-6)=0$

$\Rightarrow \quad x=-\frac{5}{2},6$

On checking, we find that $-\frac{5}{2}$ is an extraneous root.
Hence solution set = { 6 }

Exercise 4.3

Solve the following equations:

1. $3x^2+2x-\sqrt{3x^2+2x-1}=3$
2. $x^2 - - - 7 = x-3\sqrt{2x^2-3x+2}$
3. $\sqrt{2x+8}+\sqrt{x+5}=7$
4. $\sqrt{3x+4}=2+\sqrt{2x-4}$
5. $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$
6. $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$
7. $\sqrt{x^2+2x-3}+\sqrt{x^2+7x-8}=\sqrt{5(x^2+3x-4)}$
8. $\sqrt{2x^2-5x-3}+3\sqrt{2x+1}=\sqrt{2x^2+25x+12}$
9. $\sqrt{3x^2-5x+2}+\sqrt{6x^2-11x+5}=\sqrt{5x^2-9x+4}$
10. $(x+4)(x+1)=\sqrt{x^2+2x-15}+3x+31$
11. $\sqrt{3x^2-2x+9}+\sqrt{3x^2-2x-4}=13$
12. $\sqrt{5x^2+7x+2}-\sqrt{4x^2+7x+18}=x-4$

4.3 Three Cube Roots of Unity

Let x be a cube root of unity

$\therefore \quad x=\sqrt[3]{1}=(1)^{\frac{1}{3}}$

$\Rightarrow \quad x^3=1$

$\Rightarrow \quad x^3-1=0$

$\Rightarrow \quad (x-1)(x^2+x+1)=0$

Either $x-1=0 \Rightarrow x=1$

or $x^2+x+1=0$

$$\therefore \quad x=\frac{-1\pm\sqrt{1-4}}{2} \quad =\frac{-1\pm\sqrt{-3}}{2}$$

$$\Rightarrow \quad x=\frac{-1\pm\sqrt{3}i}{2} (\because \sqrt{-1}=i)$$

Thus the three cube roots of unity are:

$$1, \frac{-1+\sqrt{3}i}{2} \text{ and } \frac{-1-\sqrt{3}i}{2}$$

Note: We know that the numbers containing i are called **complex** numbers. So $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$ are called complex or imaginary cube roots of unity.

*By complex root we mean, a root containing non-zero imaginary part.

4.3.1 Properties of Cube Roots of Unity

i) Each complex cube root of unity is square of the other

Proof: (a) $\left(\frac{-1+\sqrt{3}\,i}{2}\right)^2 = \frac{(-1)^2+(\sqrt{3}\,i)^2+2(-1)(\sqrt{3}\,i)}{4}$

$$= \frac{1-3-2\sqrt{3}\,i}{4} = \frac{-2-2\sqrt{3}\,i}{4}$$

$$= 2\left(\frac{-1-\sqrt{3}\,i}{4}\right)$$

$$= \frac{-1-\sqrt{3}\,i}{2}$$

(b) $\left(\frac{-1-\sqrt{3}i}{2}\right)^2 = \left[-\frac{(1+\sqrt{3}i)}{2}\right]^2$

$$= \frac{(1)^2+(\sqrt{3}\,i)^2+(2)(1)(\sqrt{3}\,i)}{4}$$

$$= \frac{1-3+2\sqrt{3}i}{4} = \frac{-2+2\sqrt{3}i}{4}$$

$$= 2\left(\frac{-1+\sqrt{3}\,i}{4}\right)$$

$$= \frac{-1+\sqrt{3}\,i}{2}$$

Hence each complex cube root of unity is square of the other.

Note: if $\frac{-1+\sqrt{3}\,i}{2} = \omega$, then $\frac{-1-\sqrt{3}\,i}{2} = \omega^2$,

and if $\frac{-1-\sqrt{3}\,i}{2} = \omega$, then $\frac{-1+\sqrt{3}\,i}{2} = \omega^2$ [ω is read as omega]

ii) The Sum of all the three cube roots of unity is zeroi.e. $1+\omega+\omega^2 = 0$

Proof: We know that cube roots of unity are

$$1, \frac{-1+\sqrt{3}\,i}{2} \text{ and } \frac{-1-\sqrt{3}\,i}{2}$$

Sum of all the three cube roots = $1+\frac{-1+\sqrt{3}\,i}{2}+\frac{-1-\sqrt{3}\,i}{2}$

$$= \frac{2-1+\sqrt{3}\,i-1-\sqrt{3}\,i}{2} = \frac{0}{2}=0$$

if $\omega = \frac{-1+\sqrt{3}i}{2}$, then $\omega^2 = \frac{-1-\sqrt{3}i}{2}$

Hence sum of cube roots of unity $=1+\omega+\omega^2=0$

iii) The product of all the three cube roots of unity is unity i.e., $\omega^3 = 1$

Proof: Let $\dfrac{-1+\sqrt{3}i}{2} = \omega$ and $\dfrac{-1-\sqrt{3}i}{2} = \omega^2$

$$\therefore \quad 1.\omega.\omega^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$= \frac{(-1)^2-(\sqrt{3}i)^2}{4}$$

$$= \frac{1-(-3)}{4} = \frac{1+3}{4}$$

$$\Rightarrow \quad \omega^3 = 1$$

$\therefore$ Product of the complex cube roots of unity $= \omega^3 = 1.$

iv) For any $n \in z, \omega^n$ is equivalent to one of the cube roots of unity.

With the help of the fact that $\omega^3 = 1$, we can easily reduce the higher exponent of ω to its lower equivalent exponent.

e.g.
$$\omega^4 = \omega^3.\omega = 1.\omega = \omega$$
$$\omega^5 = \omega^3.\omega^2 = 1.\omega^2 = \omega^2$$
$$\omega^6 = (\omega^3)^2 = (1)^2 = 1$$
$$\omega^{15} = (\omega^3)^5 = (1)^5 = 1$$
$$\omega^{27} = (\omega^3)^9 = (1)^9 = 1$$
$$\omega^{11} = \omega^9.\omega^2 = (\omega^3)^3.\ \omega^2 = (1)^3.\omega^2 = \omega^2$$
$$\omega^{-1} = \omega^{-3}.\omega^2 = (\omega^3)^{-1}.\omega^2 = \omega^2$$
$$\omega^{-5} = \omega^{-6}.\omega = (\omega^3)^{-2}.\omega = \omega$$
$$\omega^{-12} = (\omega^3)^{-4} = (1)^{-4} = 1$$

Example 1: Prove that: $(x^3+y^3) = (x+y)(x+\omega y)(x+\omega^2 y)$

Solution :
$$\text{R.H.S} = (x+y)(x+\omega y)(x+\omega^2 y)$$
$$= (x+y)[x^2+(\omega+\omega^2)yx+\omega^3 y^2]$$
$$= (x+y)(x^2-xy+y^2) = x^3+y^3 \qquad \{\because \omega^3 = 1, \omega+\omega^2 = -1\}$$
$$= \text{L.H.S.}$$

Example 2: Prove that: $=(-1+\sqrt{-3})^4 + (-1-\sqrt{-3})^4 = -16$

Solution:
$$\text{L.H.S} = (-1+\sqrt{-3})^4 + (-1-\sqrt{-3})^4$$
$$= \left[2\left(\frac{-1+\sqrt{-3}}{2}\right)\right]^4 + \left[2\left(\frac{-1-\sqrt{-3}}{2}\right)\right]^4$$
$$= (2\omega)^4 + (2\omega^2)^4$$
$$= 16\omega^4 + 16\omega^8$$
$$= 16(\omega^4+\omega^8)$$
$$= 16\left[\omega^3.\ \omega + \omega^6.\ \omega^2\right]$$
$$= 16(\omega+\omega^2)$$
$$= 16(-1)$$
$$= -16 = \text{R.H.S}$$

$\left\{ \text{Let } \dfrac{-1+\sqrt{-3}}{2} = \omega \quad \therefore \dfrac{-1-\sqrt{-3}}{2} = \omega^2 \right.$

$\because \omega^3 = \omega^6 = 1$

$\because \omega + \omega^2 = -1$

4.4 Four Fourth Roots of Unity

Let x be the fourth root of unity

$$\therefore \quad x = \sqrt[4]{1} = (1)^{\frac{1}{4}}$$
$$\Rightarrow \quad x^4 = 1$$
$$\Rightarrow \quad x^4 - 1 = 0$$
$$\Rightarrow \quad (x^2-1)(x^2+1) = 0$$
$$\Rightarrow \quad x^2-1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$
$$\text{and } x^2+1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm i.$$

Hence four fourth roots of unity are:

$+1, -1, +i, -i.$

4.4.1 Properties of four Fourth Roots of Unity

We have found that the four fourth roots of unity are:

$+1, -1, +i, -i$

i) Sum of all the four fourth roots of unity is zero

$\because \quad +1+(-1)+i+(-i)=0$

ii) The real fourth roots of unity are additive inverses of each other

+1 and –1 are the real fourth roots of unity

and +1 + (–1) = 0 = (–1) + 1

iii) Both the complex/imaginary fourth roots of unity are conjugate of each other

i and $-i$ are complex / imaginary fourth roots of unity, which are obviously conjugates of each other.

iv) Product of all the fourth roots of unity is –1

$\therefore \quad 1\times(-1)\times i\times(-i)=-1$

Exercise 4.4

1. Find the three cube roots of: 8, – 8, 27, –27, 64.

2. Evaluate:

i) $(1+\omega-\omega^2)^8$ ii) $\omega^{28}+\omega^{29}+1$ iii) $(1+\omega-\omega^2)(1-\omega+\omega^2)$

iv) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^7$ v) $(-1+\sqrt{-3})^5+(-1-\sqrt{3})^5$

3. Show that:

i) $x^3-y^3=(x-y)(x-\omega y)(x-\omega^2 y)$

ii) $x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

iii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots.2n \text{ factors}=1$

Hint: $1+\omega^4=1+\omega^3.\omega=1+\omega=-\omega^2, 1+\omega^8=1+\omega^6.\omega^2=1+\omega^2=-\omega$

4. If ω is a root of $x^2+x+1=0$, show that its other root is ω^2 and prove that $\omega^3=1$.

5. Prove that complex cube roots of –1 are $\dfrac{1+\sqrt{3}i}{2}$ and $\dfrac{1-\sqrt{3}i}{2}$ and hence prove that

$$\left(\frac{1+\sqrt{-3}}{2}\right)^9+\left(\frac{1-\sqrt{-3}}{2}\right)^9 = 2..$$

6. If ω is a cube root of unity, form an equation whose roots are 2ω and $2\omega^2$.

7. Find four fourth roots of 16, 81, 625.

8. Solve the following equations:

i) $2x^4-32=0$ ii) $3y^5-243y=0$

iii) $x^3+x^2+x+1=0$ iv) $5x^5-5x=0$

4.5 Polynomial Function:

A polynomial in x is an expression of the form

$$a_nx^n+a_{n-1}x^{n-1}+\ldots.+a_{1n}x+a_0, \qquad a_n\neq 0 \qquad \text{(i)}$$

where n is a non-negative integer and the coefficients $a_n, a_{n-1},\ldots., a_1$ and a_0 are real numbers. It can be considered as a **Polynomial function** of x. The highest power of x in polynomial in x are called the **degree** of the polynomial. So the expression (i), is a polynomial of degree n. The polynomials x^2-2x+3, $3x^3+2x^2-5x+4$ are of degree 2 and 3 respectively.

Consider a polynomial; $3x^3-10x^2+13x-6$.

If we divide it by a linear factor $x-2$ as shown below, we get a quotient x^2-4x+5 and a remainder 4 .

$$\begin{array}{r} 3x^2 - 4x + 5 \\ \text{divisor} \rightarrow x-2 \overline{) 3x^3 - 10x^2 + 13x - 6} \leftarrow \text{dividend} \\ 3x^3 - 6x^2 \\ - \quad + \\ \hline -4x^2 + 13x \\ -4x^2 + 8x \\ + \quad - \\ \hline 5x - 6 \\ 5x - 10 \\ - \quad + \\ \hline 4 \leftarrow \text{remainder} \end{array}$$

Hence we can write: $3x^3 - 10x^2 + 13x - 6 = (x-2)(3x^2 - 4x + 5) + 4$

i.e., dividend = (divisor) (quotient) + remainder.

4.6 Theorems:

Remainder Theorem: If a polynomial $f(x)$ of degree $n \geq 1$, n is non-negative integer is divided by $x - a$ till no x-term exists in the remainder, then $f(a)$ is the remainder.

Proof: Suppose we divide a polynomial $f(x)$ by $x - a$. Then there exists a unique quotient $q(x)$ and a unique remainder R such that $f(x) = (x - a)(qx) + R$ (i)

Substituting $x = a$ in equation (i), we get

$f(a) = (a - a)q(a) + R$

$\Rightarrow$ $f(a) = R$

Hence remainder = $f(a)$

Note: Remainder obtained when $f(x)$ is divided by $x - a$ is same as the value of the polynomial $f(x)$ at $x = a$.

Example 1: Find the remainder when the polynomial $x^3 + 4x^2 - 2x + 5$ is divided by $x - 1$.

Solution: Let $f(x) = x^3 + 4x^2 - 2x + 5$ and $x - a = x - 1 \Rightarrow a = 1$

Remainder = $f(1)$ (By remainder theorem)

$= (1)^3 + 4(1)^2 - 2(1) + 5$

$= 1 + 4 - 2 + 5$

$= 8$

Example 2: Find the numerical value of k if the polynomial $x^3 + kx^2 - 7x + 6$ has a remainder of -4, when divided by $x + 2$.

Solution: Let $f(x) = x^3 + kx^2 - 7x + 6$ and $x - a = x + 2$, we have, $a = -2$

Remainder = $f(-2)$ (By remainder theorem)

$= (-2)^3 + k(-2)^2 - 7(-2) + 6$

$= -8 + 4k + 14 + 6$

$= 4k + 12$

Given that remainder = -4

$\therefore$ $4k + 12 = -4$

$\Rightarrow$ $4k = -16$

$\Rightarrow$ $k = -4$

Factor Theorem: The polynomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$ i.e.; $(x - a)$ is a factor of $f(x)$ if and only if $x = a$ is a root of the polynomial equation $f(x) = 0$.

Proof: Suppose $g(x)$ is the quotient and R is the remainder when a polynomial $f(x)$ is divided by $x - a$, then by **Remainder Theorem**

$f(x) = (x-a)\ g(x) + R$

Since $f(a) = 0$ $\Rightarrow R = 0$

$\therefore$ $f(x) = (x-a)\ g(x)$

$\therefore$ $(x-a)$ is a factor of $f(x)$.

Conversely, if $(x-a)$ is a factor of $f(x)$, then

$R = f(a) = 0$

which proves the theorem.

Note: To determine if a given linear polynomial $x - a$ is a factor of $f(x)$, all we need to check whether $f(a) = 0$.

Example 3: Show that $(x - 2)$ is a factor of $x^4 - 13x^2 + 36$.

Solution: Let $f(x) = x^4 - 13x^2 + 36$ and $x - a = x - 2 \Rightarrow a = 2$

Now $f(2) = (2)^4 - 13(2)^2 + 36$

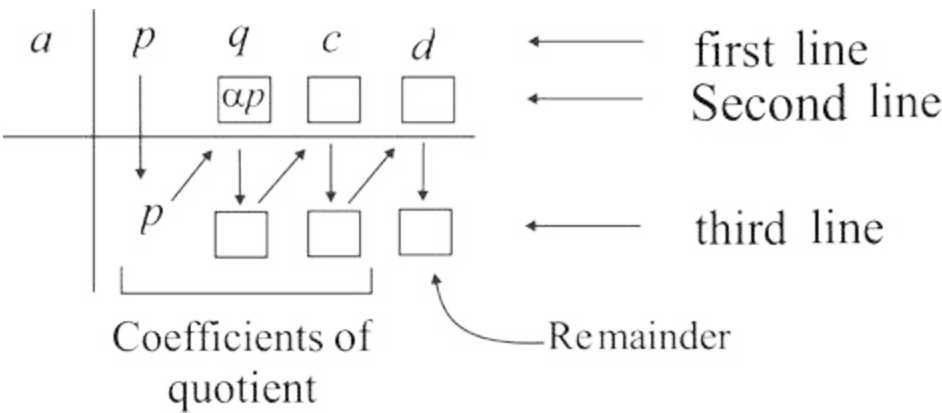
$= 16 - 52 + 36$

$= 0$ = remainder

$\Rightarrow (x - 2)$ is a factor of $x^4 - 13x^2 + 36$

4.7 Synthetic Division

There is a nice shortcut method for long division of a polynomial $f(x)$ by a polynomial of the form $x - a$. This process of division is called **Synthetic Division**.

To divide the polynomial $px^3 + qx^2 + cx + d$ by $x - a$

Out Line of the Method:

i) Write down the coefficients of the dividend $f(x)$ from left to right in decreasing order of powers of x. Insert 0 for any missing terms.

ii) To the left of the first line, write a of the divisor $(x - a)$.

iii) Use the following patterns to write the second and third lines:

Vertical pattern $(\downarrow)$ Add terms

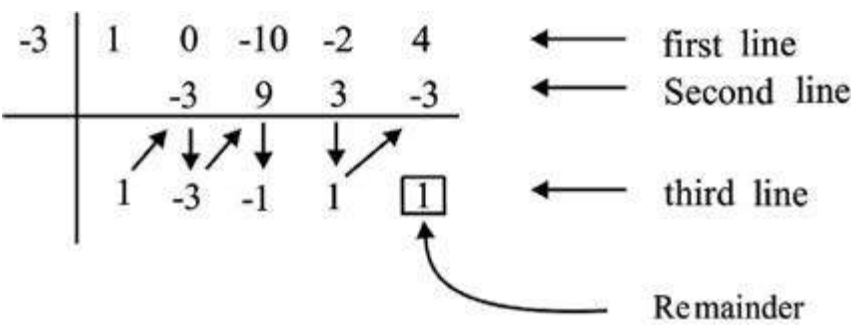
Diagonal pattern $(\nearrow)$ Multiply by a.

Example 4: Use synthetic division to find the quotient and the remainder when the polynomial $x^4 - 10x^2 - 2x + 4$ is divided by $x + 3$.

Solution: Let $f(x) = x^4 - 10x^2 - 2x + 4$

$= x^4 + 0x^3 - 10x^2 - 2x + 4$

and $x - a = x + 3 = x - (-3) \Rightarrow x = -3$

Dividend $x^4 - 10x^2 - 2x + 4$

$\therefore$ Quotient $= x^3 - 3x^2 - x + 1$

Remainder $= 1$

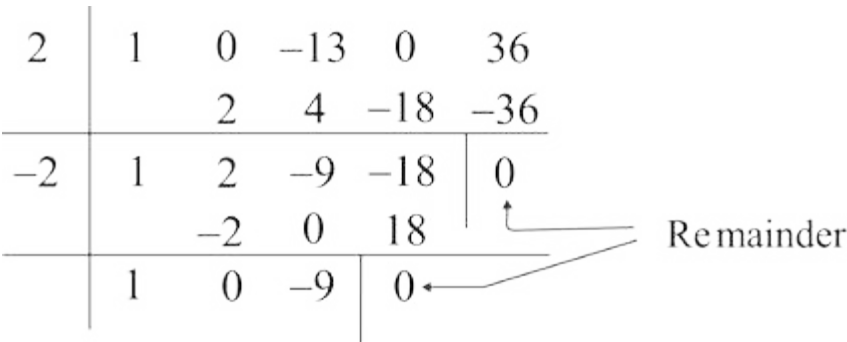
Example 5: If $(x - 2)$ and $(x + 2)$ are factors of $x^4 - 13x^2 + 36$. Using synthetic division, find the other two factors.

Solution: Let $f(x) = x^4 - 13x^2 + 36$

$= x^4 + 0x^3 - 13x^2 - 0x + 36$

Here $x - a = x - 2 \Rightarrow x = 2$ and $x - a = x + 2 = x - (-2) \Rightarrow x = -2$

By synthetic Division:

2	1	0	−13	0	36
		2	4	−18	−36
−2	1	2	−9	−18	0 ← Remainder
		−2	0	18	
	1	0	−9	0 ← Remainder	

$\therefore$ Quotient $= x^2 + 0x - 9$

$= x^2 - 9$

$= (x + 3)(x - 3)$

$\therefore$ Other two factors are $(x + 3)$ and $(x - 3)$.

Example 6: If $x+1$ and $x-2$ are factors of x^3+px^2+qx+2. By use of synthetic division find the values of p and q.

Solution: Here $x-a=x+1 \Rightarrow a=-1$ and $x-a=x-2 \Rightarrow a=2$

Let $f(x) = x^3+px^2+qx+2$

By Synthetic Division:

-1	1	p	q	2
		-1	$-p+1$	$-q+p-1$
2	1	$p-1$	$q-p+1$	$1-q+p$ ← Remainder
		2	$2p+2$	
	1	$p+1$	$p+q+3$ ← Remainder	

Since $x+1$ and $x-2$ are the factors of $f(x)$

$\therefore$ $p - q + 1 = 0$ (i)

and $p + q + 3 = 0$ (ii)

Adding (i) & (ii) we get $2p + 4 = 0$ $\Rightarrow p = -2$

from (i) $-2 - q + 1 = 0$ $\Rightarrow q = -1$

Example 7: By the use of synthetic division, solve the equation $x^4-5x^2+4=0$ if -1 and 2 are its roots.

-1	1	0	-5	0	4
		-1	1	4	-4
2	1	-1	-4	4	0 ← Remainder
		2	2	-4	
	1	1	-2	0 ← Remainder	

Solution: $f(x) = x^4 - 0x^3 - 5x^2 + 0x + 4$

Depressed Equation:

$x^2 + x - 2 = 0$

$\Rightarrow (x + 2)(x - 1) = 0$ $\Rightarrow x = -2$ or $x = 1$

Hence Solution set = $\{-2, -1, 1, 2\}$.

Exercise 4.5

Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial:

1. $x^2 + 3x + 7$, $x + 1$

2. $x^3 - x^2 + 5x + 4$, $x - 2$

3. $3x^4 + 4x^3 + x - 5$, $x + 1$

4. $x^3 - 2x^2 + 3x + 3$, $x - 3$

Use the factor theorem to determine if the first polynomial is *a* factor of the second polynomial.

5. $x - 1$, $x^2 + 4x - 5$

6. $x - 2$, $x^3 + x^2 - 7x + 1$

7. $\omega + 2$, $2\omega^3 + \omega^2 - 4\omega + 7$

8. $x - a$, $x^n - a^n$ where n is a positive integer

9. $x + a$, $x^n + a^n$ where n is an odd integer.

10. When $x^4+2x^3+kx^2+3$ is divided by $x-2$ the remainder is 1. Find the value of k.

11. When the polynomial x^3+2x^2+kx+4 is divided by $x-2$ the remainderis 14. Find the value of k.

Use Synthetic division to show that x is the solution of the polynomial and use the result to factorize the polynomial completely.

12. $x^3-7x+6=0,\ x=2$

13. $x^3-28x-48=0, -x=\ 4$

14. $2x^4+7x^3-4x^2-27x-18,\ x=2, -\ x=\ 3$

15. Use synthetic division to find the values of p and q if $x + 1$ and $x - 2$ are the factors of the polynomial x^3+px^2+qx+6.

16. Find the values of a and b if –2 and 2 are the roots of the polynomial x^3-4x^2+ax+b.

4.8 Relations Between the Roots and the Coefficients of a Quadratic Equation

Let α,β are the roots of $ax^2+bx+c=0, a\neq 0$ such that

$\frac{-b+\sqrt{b\ -\ ac}}{}$ and $\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$

$$\therefore\ \alpha+\beta = \frac{-b+\sqrt{b^2-4ac}}{2a}\ \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}\ -\ -\frac{2b}{2a}\quad \frac{b}{a}$$

and $$\alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}$$

$$=\frac{b^2-b^2+4ac}{4a^2}=\frac{4ac}{4a^2}\quad\frac{c}{a}$$

Sum of the roots = S $-\frac{b}{a}=-\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

Product of the roots = $P=\frac{c}{a}=\frac{\text{constant term}}{\text{coefficient of } x^2}$

The above results are helpful in expressing symmetric functions of the roots in terms of the coefficients of the quadratic equations.

Example 1: If α,β are the roots of $ax^2+bx+c=0, a\neq 0$, find the values of

i) $\alpha^2+\beta^2$ ii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$ iii) $(\alpha-\beta)^2$

Solution: Since α,β are the roots of $ax^2+bx+c=0$

$$\therefore\ \alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

i) $$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

$$=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ca}{a^2}$$

ii) $$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$=\frac{\left(-\frac{b}{a}\right)^3-3\frac{c}{a}\left(-\frac{b}{c}\right)}{\frac{c}{a}}\quad\frac{\frac{-b^3+3abc}{a^3}}{\frac{c}{a}}$$

$$=\frac{-b^3+3abc}{a^2c}$$

iii) $$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$$

$$=\left(-\frac{b}{a}\right)^2-4\left(\frac{c}{a}\right)=\frac{b^2}{a^2}-4\frac{c}{a}=\frac{b^2-4ac}{a^2}$$

Example 2: Find the condition that one root of $ax^2+bx+c=0, a\neq 0$ is square of the other.

Solution: As one root of $ax^2+bx+c=0$ is square of the other, let the roots be α and α^2

Sum of roots $a+a^2=-\frac{b}{a}$ (i)

Product of roots= $a\,.\,a^2=\frac{c}{a}$ $\Rightarrow \alpha^3=\frac{c}{a}$ (ii)

Cubing both sides of (i), we get

$$a^3 + a^6 + 3aa^2(a + a^2) = -\frac{b^3}{a^3}$$

$$\Rightarrow \quad a^3 + (a^3)^2 + 3a^3(a + a^2) = \frac{b^3}{a^3}$$

$$\Rightarrow \quad \frac{c}{a} + \left(\frac{c}{a}\right)^2 + 3\frac{c}{a}\left(-\frac{b}{a}\right) = \frac{b^3}{a^3} \qquad \text{(From (i), (ii))}$$

$$\Rightarrow \quad a^2c + ac^2 - 3abc = b^3$$

4.9 Formation of an Equation Whose Roots are Given

$\therefore \quad (x - \alpha)(x - \beta) = 0$ has the roots α and β

$\Rightarrow \quad x^2 - (\alpha + \beta)x + \alpha\beta = 0$ has the roots α and β.

For S = Sum of the roots and P = Product of the roots.

Thus $x^2 - Sx + P = 0$

Example 3: If α, β are the root of $ax^2 + bx + c = 0$ form the equation whose roots are double the roots of this equation.

Solution: $\because$ α and β are the root of $ax^2 + bx + c = 0$

$$\therefore \quad \alpha + \beta - = \frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

The new roots are 2α and 2β.

$\therefore$ Sum of new roots $= 2\alpha + 2\beta$

$$= 2(\alpha + \beta) = -\frac{2b}{a}$$

Product of new roots $= 2\alpha . 2\beta = 4\alpha\beta = \frac{4c}{a}$

Required equation is given by

y^2 –(Sum of roots) y + Product of roots = 0

$$\Rightarrow \quad y^2 + \frac{2b}{a}y + \frac{4c}{a} = 0 \qquad \Rightarrow \quad ay^2 + 2by + 4c = 0$$

Exercise 4.6

1. If α, β are the root of $3x^2 - 2x + 4 = 0$, find the values of

i) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ **ii)** $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ **iii)** $a^4 + \beta^4$

iv) $a^3 + \beta^3$ **v)** $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ **vi)** $a^2 - \beta^2$

2. If α, β are the root of $x^2 - px - p - c = 0$, prove that
$(1 + \alpha)(1 + \beta) = 1 - c$

3. Find the condition that one root of $x^2 + px + q = 0$ is
i) double the other **ii)** square of the other
iii) additive inverse of the other
iv) multiplicative inverse of the other.

4. If the roots of the equation $x^2 - px + q = 0$ differ by unity, prove that $p^2 = 4q + 1$.

5. Find the condition that $\frac{a}{x-a} + \frac{b}{x-b} = 5$ may have roots equal in magnitude but opposite in signs.

6. If the roots of $px^2 + qx + q = 0$ are α and β then prove that $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0$.

7. If α, β are the roots of the equation $ax^2 + bx + c = 0$, form the equations whose roots are

i) a^2, β^2 **ii)** $\frac{1}{\alpha}, \frac{1}{\beta}$ **iii)** $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$

iv) a^3, β^3 **v)** $\frac{1}{\alpha^3}, \frac{1}{\beta^3}$ **vi)** $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}$

vii) $(\alpha-\beta)^2, (\alpha+\beta)^2$ **viii)** $-\frac{1}{\alpha^3}, -\frac{1}{\beta^3}$

8. If α , β are the roots of the $5x^2 - x - 2 = 0$, form the equation whose roots are $\frac{3}{\alpha}$ and $\frac{3}{\beta}$.

9. If α , β are the roots of the $x^2 - 3x + 5 = 0$, form the equation whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$.

4.10 Nature of the roots of a quadratic equation

We know that the roots of the quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula as: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

We see that there are two possible values for x, as discriminated by the part of the formula $\pm\sqrt{b^2-4ac}$.

The nature of the roots of an equation depends on the value of the expression $b^2 - 4ac$, which is called its **Discriminant.**

Case 1: If $b^2 - 4ac = 0$ then the roots will be $-\frac{b}{2a}$ and $-\frac{b}{2a}$ So, the roots are real and repeated equal.

Case 2: If $b^2 - 4ac < 0$ then $\sqrt{b^2-4ac}$ will be imaginary
So, the roots are complex / imaginary and distinct / unequal

Case 3: If $b^2 - 4ac > 0$ then $\sqrt{b^2-4ac}$ will be real.
So, the roots are real and distinct / unequal.

However, If $b^2 - 4ac$ is a perfect square then $\sqrt{b^2-4ac}$ will be rational, and so the roots are rational, otherwise irrational.

Example 1: Discuss the nature of the roots of the following equations:

i) $x^2 + 2x + 3 = 0$ ii) $2x^2 + 5x - 1 = 0$
iii) $2x^2 - 7x + 3 = 0$ iv) $9x^2 - 12x + 4 = 0$

Solution:

i) Comparing $x^2 + 2x + 3 = 0$ with $ax^2 + bx + c = 0$, we have
$a = 1, b = 2, c = 3$
Discriminant (Disc) $= b^2 - 4ac$
$= (2)^2 - 4(1)(3) = 4 - 12 = -8$
$\Rightarrow$ Disc < 0
$\therefore$ The roots are complex / imaginary and distinct / unequal.

ii) Comparing $2x^2 + 5x - 1 = 0$ with $ax^2 + bx + c = 0$, we have
$a = 2, b = 5, c = -1$
Disc $= b^2 - 4ac$
$= (5)^2 - 4(2)(-1)$
$= 25 + 8 = 33$
$\Rightarrow$ Disc > 0 but not a perfect square.
$\therefore$ The roots are irrational and unequal.

iii) Comparing $2x^2 - 7x + 3 = 0$ with $ax^2 + bx + c = 0$ we have
$a = 2, b = -7, c = 3$
Disc $= b^2 - 4ac$
$= (-7)^2 - 4(2)(3)$
$= 49 - 24 = 25 = 5^2$
$\Rightarrow$ Disc > 0 and a perfect square.
$\therefore$ The roots are irrational and unequal.

iv) Comparing $9x^2 - 12x + 4 = 0$ with $ax^2 + bx + c = 0$,we have
$a = 9, b = -12, c = 4$
Disc $= b^2 - 4ac$
$= (-12)^2 - 4(9)(4)$
$= 144 - 144 = 0$
$\Rightarrow$ Disc $= 0$
$\therefore$ The roots are real and equal.

Example 2: For what values of m will the following equation have equal root? $(m+1)x^2+2(m+3)x+2m+3=0, m \neq -1$

Solution: Comparing the given equation with $ax^2 + bx + c = 0$

$$a=m+1, b=2(m+3), c=2m+3$$

Disc $= b^2 - 4ac$

$$=[2(m+3)]^2-4(m+1)(2m+3)$$
$$=4(m^2+6m+9)-4(2m^2+5m+3)$$
$$= 4m^2 \quad 4m \quad 24$$

The roots of the given equation will be equal, if Disc. = 0 i.e.,

if $\quad -4m^2+4m+24=0$

$\Rightarrow \quad m^2-m-6=0$

$\Rightarrow \quad (m-3)(m+2)=0 \Rightarrow m=3$ or $m=-2$

Hence if m = 3 or m = –2, the roots of the given equation will be equal.

Example 3: Show that the roots of the following equation are real

$$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$$

Also show that the roots will be equal only if $a = b = c$.

Solution: $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$

$\Rightarrow \quad x^2-ax-bx+ab+x^2-bx-cx+bc+x^2-cx-ax+ac=0$

$\Rightarrow \quad 3x^2-2(a+b+c)x+ab+bc+ca=0$

Disc $= b^2 - 4ac$

$$=[2(a+b+c)]^2-4(3)(ab+bc+ca)$$
$$=4(a^2+b^2+c^2+2ab+2bc+2ca-3ab-3bc-3ca)$$
$$=4(a^2+b^2+c^2-ab-bc-ca)$$
$$=2(2a^2+2b^2+2c^2-2ab-2bc-2ca)$$
$$=2[a^2+b^2-2ab+b^2+c^2-2bc+c^2+a^2-2ca]$$
$$=2[(a-b)^2+(b-c)^2+(c-a)^2]$$

= 2(Sum of three squares)

Thus the discriminant cannot be negative.

Hence the roots are real.

The roots will be equal, if the discriminant = 0

This is possible only if $a - b = 0$, $b - c = 0$, $c - a = 0$ i.e., if $a = b = c$.

Exercise 4.7

1. Discuss the nature of the roots of the following equations:

i) $4x^2+6x+1=0$ ii) $x^2-5x+6=0$

iii) $2x^2-5x+1=0$ iv) $25x^2-30x+9=0$

2. Show that the roots of the following equations will be real:

i) $x^2-2\left(m+\frac{1}{m}\right)x+3=0; m \bullet 0$

ii) $(b-c)x^2+(c-a)x+(a-b)=0; a,b,c \in Q$

3. Show that the roots of the following equations will be rational:

i) $(p+q)x^2-px-q=0;$ ii) $px^2-(p-q)x-q=0;$

4. For what values of m will the roots of the following equations be equal?

i) $(m+1)x^2+2(m+3)x+m+8 \quad =0$

ii) $x^2-2(1+3m)x+7(3+2m) \quad =0$

iii) $(1+m)x^2-2(1+3m)x+(1+8m) \quad =0$

5. Show that the roots of $x^2+(mx+c)^2=a^2$ will be equal, if $c^2=a^2(1+m^2)$

6. Show that the roots of $(mx + c)^2 = 4ax$ will be equal, if $c=\frac{a}{m}; m \neq 0$

7. Prove that $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}$ =1 will have equal roots, if $c^2 = a^2m^2 + b^2$; $a \neq 0$, $b \neq 0$

8. Show that the roots of the equation $(a^2 - bc)x^2 + 2(b^2 - ca)x + c^2 - ab = 0$ will be equal, if either $a^3 + b^3 + c^3 = 3abc$ or $b = 0$.

4.11 System of Two Equations Involving Two Variables

We have, so far, been solving quadratic equations in one variable. Now we shall be solving the equations in two variables, when at least one of them is quadratic. To determine

the value of two variables, we need a pair of equations.Such a pair of equations is called a **system of simultaneous equations.**

No general rule for the solution of such equations can be laid down except that some how or the other, one of the variables is eliminated and the resulting equation in one variable is solved.

Case I: One Linear Equation and one Quadratic Equation

If one of the equations is linear, we can find the value of one variable in terms of the other variable from linear equation. Substituting this value of one variable in the quadratic equation, we can solve it. The procedure is illustrated through the following examples:

Example 1: Solve the system of equations:

$x + y = 7$ and $x^2 - xy + y^2 = 13$

Solution: $x + y = 7 \quad \Rightarrow x = 7 - y \qquad$ (i)

Substituting the value of x in the equation $x^2 - xy + y^2 = 13$ we have

$(7-y)^2 - y(7-y) + y^2 = 13$

$\Rightarrow \quad 49 - 14y + y^2 - 7y + y^2 + y^2 = 13$

$\Rightarrow \quad 3y^2 - 21y + 36 = 0$

$\Rightarrow \quad y^2 - 7y + 12 = 0$

$\Rightarrow \quad (y-3)(y-4) = 0$

$\Rightarrow \quad y = 3$ or $y = 4$

Putting $y = 3$, in (i), we get $x = 7 - 3 = 4$

Putting $y = 4$, in (i), we get $= 7 - 4 = 3$

Hence solution set = {(4, 3), (3, 4)}.

Note: Two quadratic equations in which xy term is missing and the coefficients of x^2 and y^2 are equal, give a linear equation by subtraction.

Example 2: Solve the following equations:

$x^2 + y^2 + 4x = 1$ and $x^2 + (y-1)^2 = 10$

Solution: The given system of equations is

$$\begin{cases} x^2 + y^2 + 4x = 1 & \text{(i)} \\ x^2 + y^2 - 2y + 1 = 10 & \text{(ii)} \end{cases}$$

Subtraction gives,

$4x + 2y + 8 = 0$

$\Rightarrow 2x + y + 4 = 0$

$\Rightarrow y = -2x - 4 \qquad$ (iii)

Putting the value of y in equation (i),

$x^2 + (-2x-4)^2 + 4x = 1 \Rightarrow x^2 + 4x^2 + 16x + 16 + 4x = 1$

$\Rightarrow \quad 5x^2 + 20x + 15 = 0 \qquad \Rightarrow \quad x \;\; + 4x + 3 = 0$

$\Rightarrow \quad (x+3)(x+1) = 0 \qquad \Rightarrow \quad x = -3$ or $x = -1$

Putting $x = -3$ in (iii), we get; $y = -2(-3) - 4 = 6 - 4 = 2$

Putting $x = -1$ in (iii), we get; $y = -2(-1) - 4 = 2 - 4 = -2$

Hence solution set = $\{(-3, 2), (-1, -2)\}$.

Exercise 4.8

Solve the following systems of equations:

1. $2x - y = 4; \quad 2x^2 - 4xy - y^2 = 6$
2. $x + y = 5$; $x^2 + 2y^2 = 17$
3. $3x + 2y = 7; \quad 3x^2 = 25 + 2y^2$
4. $x + y = 5; \dfrac{2}{x} + \dfrac{3}{y} = 2, x \neq 0, y \neq 0$
5. $x + y = a + b; \dfrac{a}{x} + \dfrac{b}{y} = 2$
6. $3x + 4y = 25; \quad \dfrac{3}{x} + \dfrac{4}{y} = 2$
7. $(x-3)^2 + y^2 = 5; \qquad 2x = y + 6$

8. $(x+3)^2+(y-1)^2=5;$ $\quad x^2+y^2+2x=9$

9. $x^2+(y+1)^2=18;$ $\quad (x+2)^2+y^2=21$

10. $x^2+y^2+6x=1$; $\quad x^2+y^2+2(x+y)=3$

Case II: Both the Equations are Quadratic in two Variables

The equations in this case are classified as:

i) Both the equations contain only x^2 and y^2 terms.

ii) One of the equations is homogeneous in x and y.

iii) Both the equations are non-homogeneous.

The methods of solving these types of equations are explained through the following examples:

Example 1: Solve the equations: $\begin{cases} x^2+y^2 = 25 \\ 2x^2+3y^2=6 \end{cases}$

Solution: Let $x^2=u$ and $y^2=v$

By this substitution the given equations become

$u+v=25$ (i)

$2u+3v=66$ (ii)

Multiplying both sides of the equation (i) by 2, we have

$2u+2v=50$ (iii)

Subtraction of (iii) from (ii) gives,

$v=16$

Putting the value of v in (i), we have

$u+16=25 \Rightarrow u=9$

$\therefore$ $x^2=9 \Rightarrow x=\pm 3$ and $y^2=16 \Rightarrow y=\pm 4$

Hence solution set = $\{(\pm 3, \pm 4)\}$.

Example 2: Solve the equations: $x^2-3xy+2y^2=0; 2x^2-3x+y^2=24$

Solution: The given equations are:

$x^2-3xy+2y^2=0$ (i)

$2x^2-3x+y^2=24$ (ii)

Equation $x^2-3xy+2y^2=0$ is homogeneous in x and y

$\Rightarrow (x-y)(x-2y)=0.$ (Factorizing)

$\Rightarrow x-y=0$ or $x-2y=0$

$\Rightarrow x=y$ ...(iii)

Putting the value of x in (ii), we get

$2y^2-3y+y^2=24$

$\Rightarrow y^2-y-8=0$

$\Rightarrow y=\frac{1\pm\sqrt{1+32}}{2}$

$\Rightarrow y=\frac{1\pm\sqrt{33}}{2}$

when $y=\frac{1+\sqrt{33}}{2}$

from (iii) $x=\frac{1+\sqrt{33}}{2}$

when $y=\frac{1-\sqrt{33}}{2}$

from (iii) $x=\frac{1-\sqrt{33}}{2}$

$\Rightarrow x=2y$ (iv)

Putting the value of x in (ii), we get

$2(2y)^2-3(2y)+y^2=24$

$\Rightarrow 8y^2-6y+y^2=24$

$\Rightarrow 3y^2-2y-8=0$

$\Rightarrow (3y+4)(y-2)=0$

$\Rightarrow y=-\frac{4}{3}, 2$

when $y=-\frac{4}{3}$,

from (iv) $x=2\left(-\frac{4}{3}\right)=\frac{8}{3}$

when $y=2$,

from (iv) $x=2(2)=4$

Hence following is the solution set.

$$\left\{\left(\frac{1+\sqrt{33}}{2},\frac{1+\sqrt{33}}{2}\right),\left(\frac{1-\sqrt{33}}{2},\frac{1-\sqrt{33}}{2}\right),\left(-\frac{8}{3},-\frac{4}{3}\right)(4,2)\right\}$$

Example 3: Solve the equations:

$$\begin{cases} x^2-y^2 = 5 \\ 4x^2-3xy=18 \end{cases}$$

Solution Given that $\begin{cases} x^2 - y^2 \quad = 5 & \text{(i)} \\ 4x^2 - 3xy = 18 & \text{(ii)} \end{cases}$

We can get a homogeneous equation in x and y, if we get rid of the constants. For the purpose, we multiply both sides of equation (i) by 18 and both sides of equation (ii) by 5 and get

$$\begin{cases} 18x^2 - 18y^2 = 90 \\ 20x^2 - 15xy = 90 \end{cases}$$

Subtraction gives,

$$2x^2 - 15xy + 18y^2 = 0$$

$$\Rightarrow \quad (x - 6y)(2x - 3y) = 0$$

$$\Rightarrow \quad x - 6y = 0 \text{ or } 2x - 3y = 0$$

Combining each of these equations with any one of the given equations, we can solve them by the method used in the example 1.

$x - 6y = 0$

$\Rightarrow \quad x = 6y$

$\because \quad x^2 - y^2 = 5$ from (i)

$\therefore \quad (6y)^2 - y^2 = 5$

$\Rightarrow \quad 35y^2 = 5$

$\Rightarrow \quad y^2 = \frac{1}{7}$

$\Rightarrow \quad y = \pm\frac{1}{\sqrt{7}}$

when $y = \frac{1}{\sqrt{7}}$,

when $\cdot y = \frac{1}{\sqrt{7}} x \quad 6\left(\frac{-1}{\sqrt{7}}\right) \quad \frac{-6}{\sqrt{7}}$

or

$2x - 3y = 0$

$\Rightarrow \quad 2x = 3y \quad \Rightarrow \quad x = \frac{3}{2}y$

$\because \quad x^2 - y^2 = 5$ from (i)

$\therefore \quad \left(\frac{3}{2}y\right)^2 - y^2 = 5$

$\Rightarrow \quad 9y^2 - 4y^2 = 20$

$\Rightarrow \quad 5y^2 = 20$

$\Rightarrow \quad y^2 = 4$

$\Rightarrow \quad y = \pm 2$

when $y = 2$,

when $y = -2$

$x = \frac{3}{2}(-2) = -3$

Hence Solution set $= \left\{\left(\frac{6}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right), \left(\frac{6}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right), (3, 2), (3,2)\right\}$

Exercise 4.9

Solve the following systems of Equations:

1. $2x^2 = 6 + 3y^2$; $3x^2 - 5y^2 = 7$
2. $8x^2 = y^2$; $x^2 + 2y^2 = 19$
3. $2x^2 - 8 = 5y^2$; $x^2 - 13 - = 2y^2$
4. $x^2 - 5xy + 6y^2 = 0$; $x^2 + y^2 = 45$
5. $12x^2 - 25xy + 12y^2 = 0$; $4x^2 + 7y^2 = 148$
6. $12x^2 - 11xy + 2y^2 = 0$; $2x^2 + 7xy = 60$
7. $x^2 - y^2 = 16$; $xy = 15$
8. $x^2 + xy = 9$; $x^2 - y^2 = 2$
9. $y^2 - 7 = 2xy$; $2x^2 + 3 = xy$
10. $x^2 + y^2 = 5$; $xy = 2$

4.12 Problems on Quadratic Equations

We shall now proceed to solve the problems which, when expressed symbolically, lead to quadratic equations in one or two variables.

In order to solve such problems, we must:

1) Suppose the unknown quantities to be x or y etc.
2) Translate the problem into symbols and form the equations satisfying the given conditions.

Translation into symbolic expression is the main feature of solving problems leading to equations. So, it is always helpful to proceed from concrete to abstract e.g. we may say that:

i) 5 is greater than 3 by 2 = 5 – 3 **ii)** x is greater than 3 by $x-3$
iii) 5 is greater than y by $5-y$ **iv)** x is greater than y by $x-y$.

The method of solving the problems will be illustrated through the following examples:

Example 1: Divide 12 into two parts such that the sum of their squares is greater than twice their product by 4.

Solution: Suppose one part = x

$\therefore$ The other part = $12-x$

Sum of the squares of the parts = $x^2+(12-x)^2$

twice the product of the parts = $2(x)(12-x)$

By the condition of the question,

$$x^2+(12-x)^2-2x(12-x)=4$$

$$\Rightarrow x^2+144-24x+x^2-24x+2x^2=4$$

$$\Rightarrow 4x^2-48x+140=0 \qquad \Rightarrow x^2-12x+35=0$$

$$\Rightarrow (x-5)(x-7)=0 \qquad \Rightarrow x=5 \text{ or } x=7$$

If one part is 5, then the other part = 12 – 5 = 7,
and if one part is 7, then the other part = 12 – 7 = 5

Here both values of x are admissible.
Hence required parts are 5 and 7.

Example 2: A man distributed Rs.1000 equally among destitutes of his street. Had there been 5 more destitutes each one would have received Rs. 10 less. Find the number of destitutes.

Solution: Suppose number of destitutes = x

Total sum = 1000 Rs.

$\therefore$ Each desitute gets = $\frac{1000}{x}$ Rs.

For 5 more destitutes, the number of destitutes would have been $x+5$

$\therefore$ Each destitute would have got = $\frac{1000}{x+5}$ Rs.

This sum would have been Rs. 10 less than the share of each destitute in the previous case.

$$\therefore \frac{1000}{x+5}=\frac{1000}{x}-10$$

$$\Rightarrow 1000x = 1000(x+5)-10(x+5)(x)$$

$$\Rightarrow x^2+5x-500=0$$

$$\Rightarrow (x+25)(x-20)=0$$

$$\Rightarrow x=-25 \text{ or } x=20$$

The number of destitutes cannot be negative. So, –25 is not admissible.
Hence the number of destitutes is 20.

Example 3: The length of a room is 3 meters greater than its breadth. If the area of the room is 180 square meters, find length and the breadth of the room.

Solution: Let the breadth of room = x meters

and the length of room = $x + 3$ meters

$\therefore$ Area of the room = $x(x + 3)$ square meters

By the condition of the question

$x(x + 3) = 180$ (i)

$\Rightarrow$ $x^2 + 3x - 180 = 0$ (ii)

$\Rightarrow$ $(x + 15)(x - 12) = 0$

$\therefore$ $x = -15$ or $x = 12$

As breadth cannot be negative so $x = -15$ is not admissible

$\therefore$ when $x = 12$, we get length $x + 3 = 12 + 3 = 15$

$\therefore$ breadth of the room = 12 meter and length of the room = 15 meter

Example 4: A number consists of two digits whose product is 8. If the digits are interchanged, the resulting number will exceed the original one by 18. Find the number.

Solution : Suppose tens digit = x

and units digit = y

$\therefore$ The number = $10x + y$

By interchanging the digits, the new number = $10y + x$

Product of the digits = xy

By the condition of question;

$xy = 8$ (i)

and $10y + x = 10x + y + 18$ (ii)

Solving (i) and (ii) ;we get

$x = -4$ or $x = 2$.

when $x = -4$, $y = -2$ and when $x = 2$, $y = 4$

Rejecting negative values of the digits,

Tens digit = 2

and Units digit = 4

Hence the required number = 24

Exercise 4.10

1. The product of one less than a certain positive number and two less than three times the number is 14. Find the number.
2. The sum of a positive number and its square is 380. Find the number.
3. Divide 40 into two parts such that the sum of their squares is greater than 2 times their product by 100.
4. The sum of a positive number and its reciprocal is $\frac{26}{5}$ .Find the number.
5. A number exceeds its square root by 56. Find the number.
6. Find two consecutive numbers, whose product is 132.
   (**Hint:** Suppose the numbers are x and $x + 1$).
7. The difference between the cubes of two consecutive even

numbers is 296. Find them.
(**Hint:** Let two consecutive even numbers be x and $x+2$)

8. A farmer bought some sheep for Rs. 9000. If he had paid Rs. 100 less for each, he would have got 3 sheep more for the same money. How many sheep did he buy, when the rate in each case is uniform?

9. A man sold his stock of eggs for Rs. 240. If he had 2 dozen more, he would have got the same money by selling the whole for Rs. 0.50 per dozen cheaper. How many dozen eggs did he sell?

10. A cyclist travelled 48 km at a uniform speed. Had he travelled 2 km/hour slower, he would have taken 2 hours more to perform the journey. How long did he take to cover 48 km?

11. The area of a rectangular field is 297 square meters. Had it been 3 meters longer and one meter shorter, the area would have been 3 square meters more. Find its length and breadth.

12. The length of a rectangular piece of paper exceeds its breadth by 5 cm. If a strip 0.5 cm wide be cut all around the piece of paper, the area of the remaining part would be 500 square cms. Find its original dimensions.

13. A number consists of two digits whose product is 18. If the digits are interchanged, the new number becomes 27 less than the original number. Find the number.

14. A number consists of two digits whose product is 14. If the digits are interchanged, the resulting number will exceed the original number by 45. Find the number.

15. The area of a right triangle is 210 square meters. If its hypoteneuse is 37 meters long. Find the length of the base and the altitude.

16. The area of a rectangle is 1680 square meters. If its diagonal is 58 meters long, find the length and the breadth of the rectangle.

17. To do a piece of work, A takes 10 days more than B. Together they finish the work in 12 days. How long would B take to finish it alone?
Hint: If some one takes x days to finish a work. The one day's work will be $\frac{1}{x}$.

18. To complete a job, A and B take 4 days working together. A alone takes twice as long as B alone to finish the same job. How long would each one alone take to do the job?

19. An open box is to be made from a square piece of tin by cutting a piece 2 dm square from each corner and then folding the sides of the remaining piece. If the capacity of the box is to be finish 128 c.dm, find the length of the side of the piece.

20. A man invests Rs. 100,000 in two companies. His total profit is Rs. 3080. If he receives Rs. 1980 from one company and at the rate 1% more from the other, find the amount of each investment.

Partial Fractions

5.1 Introduction

We have learnt in the previous classes how to add two or more rational fractions into a single rational fraction. For example,

i) $\frac{1}{x-1}+\frac{2}{x+2} = \frac{3x}{(x-1)(x+2)}$

and ii) $\frac{2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x-2}=\frac{5x^2+5x-3}{(x+1)^2(x-2)}$

In this chapter we shall learn how to reverse the order in (i) and (ii) that is to express a single rational function as a sum of two or more single rational functions which are called **Partial Fractions.**

Expressing a rational function as a sum of partial fractions is called **Partial Fraction Resolution**. It is an extremely valuable tool in the study of calculus.

An open sentence formed by using the sign of equality '=' is called an equation. The equations can be divided into the following two kinds:

Conditional equation: It is an equation in which two algebraic expressions are equal for particular value/s of the variable e.g.,

a) $2x = 3$ is a conditional equation and it is true only if $x = \frac{3}{2}$.

b) $x^2 + x - 6 = 0$ is a conditional equation and it is true for $x = 2, -3$ only.

Note: For simplicity, a conditional equation is called an **equation**.

Identity: It is an equation which holds good for all values of the variable e.g.,

a) $(a + b)x = ax + bx$ is an **identity** and its two sides are equal for all values of x.

b) $(x + 3)(x + 4) = x^2 + 7x + 12$ is also an identity which is true for all values of x.

For convenience, the symbol "=" shall be used both for equation and identity.

5.2 Rational Fraction

We know that $\frac{p}{q}$ where $p, q \in Z$ and $q \neq 0$ is called a rational number.

Similarly, the quotient of two polynomials $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$, with no common factors, is called a **Rational Fraction**. A rational fraction is of two types:

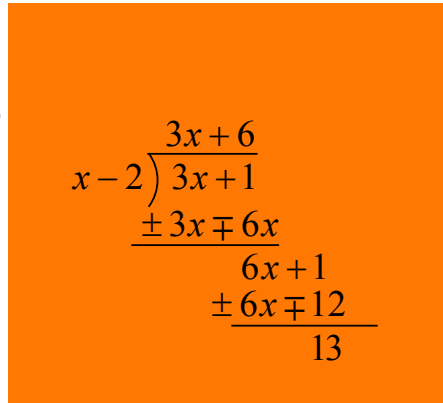
5.2.1 Proper Rational Fraction

A rational fraction $\frac{P(x)}{Q(x)}$ is called a **Proper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is less than the degree of the polynomial $Q(x)$ in the denominator. For example, $\frac{3}{x+1}, \frac{2x-5}{x^2+4}$ and $\frac{9x^2}{x^3-1}$ are proper rational fractions or proper frations.

5.2.2 Improper Rational Fraction

A rational fraction $\frac{P(x)}{Q(x)}$ is called an **Improper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is equal to or greater than the degree of the polynomial $Q(x)$ in the denominator.

For example, $\frac{x}{2x-3}, \frac{(x-2)(x+1)}{(x-1)(x+4)}, \frac{x^2-3}{3x+1}$ and $\frac{x^3-x^2+x+1}{x^2+5}$

are improper rational fractions or improper fractions.

Any improper rational fraction can be reduced by division to a mixed form, consisting of the sum of a polynomial and a proper rational fraction.

For example, $\frac{3x^2+1}{x-2}$ is an improper rational fraction. By long division we obtain $\frac{3x^2+1}{x-2} = 3x+6+\frac{13}{x-2}$

i.e., an improper rational fraction has $\frac{3x^2+1}{x-2}$ been reduced to the sum o f a polynomial $3x + 6$ and a proper rational fraction $\frac{13}{x-2}$

$$\begin{array}{r|l} & 3x+6 \\ x-2 & 3x^2+1 \\ & \pm 3x^2 \mp 6x \\ & \quad 6x+1 \\ & \pm 6x \mp 12 \\ & \quad 13 \end{array}$$

When a rational fraction is separated into partial fractions, the result is an identity; i.e., it is true for all values of the variable.

The evaluation of the coefficients of the partial fractions is based on the following theorem:

"If two polynomials are equal for all values o f the variable, then the polynomials have same degree and the coefficients of like powers of the variable in both the polynomials must be equal".

For example,

If $px^3 + qx^2 - ax + b = 2x^3 - 3x^2 - 4x + 5, \quad \forall x$

then $p = 2$, $q = -3$, $a = 4$ and $b = 5$.

5.3 Resolution of a Rational Fraction $\frac{P(x)}{Q(x)}$ into Partial Fractions

Following are the main points of resolving a rational fraction $\frac{P(x)}{Q(x)}$ into partial fractions:

i) The degree of $p(x)$ must be less than that of $Q(x)$. If not, divide and work with the remainder theorem.

ii) Clear the given equation of fractions.

iii) Equate the coefficients of like terms (powers of x).

iv) Solve the resulting equations for the coefficients.

We now discuss the following cases of partial fractions resolution.

Case I: Resolution of $\frac{P(x)}{Q(x)}$ into partial fractions when $Q(x)$ has only non-repeated linear factors:

The polynomial $Q(x)$ may be written as:

$Q(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$, where $a_1 \neq a_2 \neq \ldots \neq a_n$

$\therefore \quad \frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots\ldots + \frac{A_n}{x-a_n}$ is an identity.

Where, the coefficients $A_1, A_2, \ldots, A_n$ arenumbers to be found.

The method is explained by the following examples:

Example 1: Resolve, $\frac{7x+25}{(x+3)(x+4)}$ into Partial Fractions.

Solution: Suppose $\frac{7x+25}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$

Multiplying both sides by $(x + 3)(x + 4)$, we get

$7x + 25 = A(x + 4) + B(x + 3)$

$\Rightarrow \quad 7x + 25 = Ax + 4A + Bx + 3B$

$\Rightarrow \quad 7x + 25 = (A + B)x + 4A + 3B$

This is an identity in x.

So, equating the coefficients of like powers of x we have

$7 = A + B$ and $25 = 4A + 3B$

Solving these equations, we get $\boxed{A=4}$ and $\boxed{B=3}$.

Hence the partial fractions are: $\frac{4}{x+3} + \frac{3}{x+4}$

Alternative Method:

Suppose $\frac{7x+25}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$

$\Rightarrow \quad 7x + 25 = A(x + 4) + B(x + 3)$

As two sides of the identity are equal for all values of x,
let us put $x = -3$, and $x = -4$ in it.
Putting $x = -3$, we get $-21 + 25 = A(-3 + 4)$

$\Rightarrow \quad \boxed{A = 4}$

Putting $x = -4$, we get $-28 + 25 = B(-4 + 3)$

$\Rightarrow \quad \boxed{B = 3}$

Hence the partial fractions are: $\dfrac{4}{x+3} + \dfrac{3}{x+4}$

Example 2: Resolve $\dfrac{x^2 - 10x + 13}{(x-1)(x^2 - 5x + 6)}$ into Partial Fractions.

Solution: The factor $x^2 - 5x + 6$ in the denominator can be factorized and its factors are $x - 3$ and $x - 2$.

$$\therefore \quad \frac{x^2 - 10x + 13}{(x-1)(x^2-5x+6)} = \frac{x^2-10x+13}{(x-1)(x-2)(x-3)}$$

Suppose $\dfrac{x^2-10x+13}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$

$\Rightarrow \quad x^2 - 10x + 13 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

which is an identity in x.

Putting $x = 1$ in the identity, we get

$(1)^2 - 10(1) + 13 = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$

$\Rightarrow \quad 1 - 10 + 13 = A(-1)(-2) + B(0)(-2) + C(0)(-1)$

$4 = 2A \quad \boxed{\therefore A = 2}$

Putting $x = 2$ in the identity, we get

$(2)^2 - 10(2) + 13 = A(0)(2-3) + B(2-1)(2-3) + C(2-1)(0)$

$\Rightarrow \quad 4 - 20 + 13 = B(1)(-1)$

$\Rightarrow \quad -3 = -B \quad \boxed{\therefore B = 3}$

Putting $x = 3$ in the identity, we get

$(3)^2 - 10(3) + 13 = A(3-2)(0) + B(3-1)(0) + C(3-1)(3-2)$

$\Rightarrow \quad 9 - 30 + 13 = C(2)(1)$

$\Rightarrow \quad -8 = 2C \quad \boxed{\therefore C = -4}$

Hence partial fractions are: $\dfrac{2}{x-1} + \dfrac{3}{x-2} - \dfrac{4}{x-3}$

Note: In the solution of examples 1 and 2. We observe that the value of the constants have been found by substituting those values of x in the identities which can be got by putting each linear factor of the denominators equal to zero.

In the Example 2

a) the denominator of A is $x - 1$, and the value of A has been found by putting $x - 1 = 0$ i.e ;$x = 1$;

b) the denominator of B is $x - 2$, and the value of B has been found by putting $x - 2 = 0$ i.e., $x = 2$; and

c) the denominator of C is $x - 3$, and the value of C has been found by putting $x - 3 = 0$ i.e.,$x = 3$.

Example 3: Resolve $\dfrac{2x^3 + x^2 - x - 3}{x(2x+3)(x-1)}$ into Partial Fractions.

Solution:

$\because \dfrac{2x^3 + x^2 - x - 3}{x(2x+3)(x-1)}$ is an improper fraction so, transform it into mixed from.

Denominator $= x(2x + 3)(x - 1)$
$= 2x^3 + x^2 - 3x$

$$\begin{array}{r|l} & 1 \\ 2x^3 + x^2 - 3x & 2x^3 + x^2 - x - 3 \\ & \pm 2x^3 \pm x^2 \mp 3x \\ \hline & 2x - 3 \end{array}$$

$\therefore$ Dividing $2x^3 + x^2 - x - 3$ by $2x^3 + x^2 - 3x$, we have
Quotient = 1 and Remainder = $2x - 3$

$$\therefore \quad \frac{2x^3 + x^2 - x - 3}{x(2x+3)(x-1)} = 1 + \frac{2x-3}{x(2x+3)(x-1)}$$

Suppose $\dfrac{2x-3}{x(2x+3)(x-1)} = \dfrac{A}{x} + \dfrac{B}{2x+3} + \dfrac{c}{x-1}$

$\Rightarrow$ $2x - 3 = A(2x + 3)(x - 1) + B(x)(x - 1) + C(x)(2x + 3)$
which is an identity in x.

Putting $x = 0$ in the identity, we get $\boxed{A=1}$

Putting $2x + 3 = 0 \Rightarrow x = -\dfrac{3}{2}$ in the identity, we get $\boxed{B=-\dfrac{8}{5}}$

Putting $x - 1 = 0 \Rightarrow x = 1$ in the identity, we get $\boxed{C=-\dfrac{1}{5}}$

Hence partial fractions are: $1+\dfrac{1}{x}-\dfrac{8}{5(2x+3)}-\dfrac{1}{5(x-1)}$

Exercise 5.1

Resolve the following into Partial Fractions:

1. $\dfrac{1}{x^2-1}$
2. $\dfrac{x^2+1}{(x+1)(x-1)}$
3. $\dfrac{2x+1}{(x-1)(x+2)(x+3)}$
4. $\dfrac{3x^2-4x-5}{(x-2)(x^2+7x+10)}$
5. $\dfrac{1}{(x-1)(2x-1)(3x-1)}$
6. $\dfrac{x}{(x-a)(x-b)(x-c)}$
7. $\dfrac{6x^3+5x^2-7}{2x^2-x-1}$
8. $\dfrac{2x^3+x^2-5x+3}{2x^3+x^2-3x}$
9. $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$
10. $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$

11. $\dfrac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}$

[**Hint:** Put $x^2 = y$ to make factors of the denominator linear]

Case II: when $Q(x)$ has repeated linear factors:

If the polynomial has a factor $(x - a)^n$, $n \geq 2$ and n is a +*ve* integer, then may be written as the following identity:

$$\therefore \quad \frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)^2} + \dots\dots + \frac{A_n}{(x-a_n)^n}$$

where the coefficients $A_1, A_2, \dots, A_n$ are numbers to be found.
The method is explained by the following examples:

Example 1: Resolve, $\dfrac{x^2+x-1}{(x+2)^3}$ into partial fractions.

Solution: Suppose $\dfrac{x^2+x-1}{(x+2)^3} = \dfrac{A}{x+2}+\dfrac{B}{(x+2)^2}+\dfrac{C}{(x+2)^3}$

$\Rightarrow \quad x^2 + x - 1 = A(x + 2)^2 + B(x + 2) + C \qquad$ (i)

$\Rightarrow \quad x^2 + x - 1 = A(x^2 + 4x + 4) + B(x + 2) + C \qquad$ (ii)

Putting $x + 2 = 0$ in (i), we get

$(-2)^2 + (-2) - 1 = A(0) + B(0) + C$

$\Rightarrow \quad \boxed{1=C}$

Equating the coefficients of x^2 and x in (ii), we get $\boxed{A=1}$

and $\quad 1 = 4A + B$

$\Rightarrow \quad 1 = 4 + B \quad \Rightarrow \quad \boxed{B=-3}$

Hence the partial fractions are: $\dfrac{1}{x+2}-\dfrac{3}{(x+2)^2}+\dfrac{1}{(x+2)^3}$

Example 2: Resolve $\dfrac{1}{(x+1)^2(x^2-1)}$ into Partial Fractions.

Solution: Here denominator = $(x + 1)^2 (x^2 - 1)$

$= (x + 1)^2 (x + 1) (x - 1) = (x + 1)^3 (x - 1)$

$$\therefore \frac{1}{(x+1)^2(x^2-1)} = \frac{1}{(x+1)^3(x-1)}$$

Suppose $$\frac{}{(x-1)(x+1)^3} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

$\Rightarrow$ $1 = A(x + 1)^3 + B(x + 1)^2(x - 1) + C(x - 1)(x + 1) + D(x - 1)$ (i)

$\Rightarrow$ $1 = A(x^3 + 3x^2 + 3x + 1) + B(x^3 + x^2 - x - 1) + C(x^2 - 1) + D(x - 1)$

$\Rightarrow$ $1 = (A + B)x^3 + (3A + B + C)x^2 + (3A - B + D)x + (A - B - C - D)$ (ii)

Putting $x - 1 = 0 \Rightarrow x = 1$ in (i), we get,

$$1 = A(2)^3 \quad \Rightarrow \quad \boxed{A = \frac{1}{8}}$$

Putting $x + 1 = 0 \Rightarrow x = -1$ in (i), we get,

$$1 = D(-1 - 1) \quad \Rightarrow \quad \boxed{D = -\frac{1}{2}}$$

Equating the coefficients of x^3 and x^2 in (ii), we get

$$0 = A + B \quad \Rightarrow B = -A \quad \Rightarrow \quad \boxed{B = -\frac{1}{8}}$$

and $$0 = 3A + B + C \Rightarrow 0 = \frac{3}{8} - \frac{1}{8} + C \quad \Rightarrow \quad \boxed{C = -\frac{1}{4}}$$

Hence the partial fractions are:

$$\frac{\frac{1}{8}}{x-1} + \frac{-\frac{1}{8}}{x+1} + \frac{-\frac{1}{4}}{(x+1)^2} + \frac{-\frac{1}{2}}{(x+1)^3} = \frac{1}{8(x-1)} - \frac{1}{8(x+1)} - \frac{1}{4(x+1)^2} - \frac{1}{2(x+1)^3}$$

Exercise 5.2

Resolve the following into Partial Fractions:

1. $\dfrac{2x^2 - 3x + 4}{(x-1)^3}$ **2.** $\dfrac{5x^2 - 2x + 3}{(x+2)^3}$ **3.** $\dfrac{4x}{(x+1)^2(x-1)}$

4. $\dfrac{9}{(x+2)^2(x-1)}$ **5.** $\dfrac{1}{(x-3)^2(x+1)}$ **6.** $\dfrac{x^2}{(x-2)(x-1)^2}$

7. $\dfrac{1}{(x-1)^2(x+1)}$ **8.** $\dfrac{x^2}{(x-1)^3(x+1)}$ **9.** $\dfrac{x-1}{(x-2)(x+1)^3}$

10. $\dfrac{}{(x^2-1)(x+1)^2}$ **11.** $\dfrac{2x+1}{(x+3)(x-1)(x+2)^2}$ **12.** $\dfrac{2x^4}{(x-3)(x+2)^2}$

Case III: when $Q(x)$ contains non-repeated irreducible quadratic factor

Definition: A quadratic, factor is irreducible if it cannot be written as the product of two linear factors with real coefficients. For example, $x^2 + x + 1$ and $x^2 + 3$ are irreducible quadratic factors.

If the polynomial $Q(x)$ contains non-repeated irreducible quadratic factor then $\dfrac{P(x)}{Q(x)}$ may be written as the identity having partial fractions of the form:

$\dfrac{Ax+B}{ax^2+bx+c}$ where A and B the numbers to be found.

The method is explained by the following examples:

Example 1: Resolve $\dfrac{3x-11}{(x^2+1)(x+3)}$ into Partial Fractions.

Solution: Suppose $\dfrac{3x-11}{(x^2+1)(x+3)} = \dfrac{Ax+B}{(x^2+1)} + \dfrac{C}{(x+3)}$

$\Rightarrow \quad 3x - 11 = (Ax + B)(x + 3) + C(x^2 + 1) \quad$ (i)

$\Rightarrow \quad 3x - 11 = (A + C)x^2 + (3A + B)x + (3B + C) \quad$ (ii)

Putting $x + 3 = 0 \Rightarrow x = -3$ in (i), we get

$-9 - 11 = C(9 + 1) \Rightarrow \boxed{C = -2}$

Equating the coefficients of x^2 and x in (ii), we get

$0 = A + C \Rightarrow A = -C \Rightarrow \boxed{A = 2}$

and $3 = 3A + B \Rightarrow B = 3 - 3A \Rightarrow B = 3 - 6 \Rightarrow \boxed{B = -3}$

Hence the partial fraction are: $\dfrac{2x-3}{x^2+1} - \dfrac{2}{x+3}$

Example 2: Resolve $\dfrac{4x^2+8x}{x^4+2x^2+9}$ into Partial Fractions.

Solution: Here, denominator = $x^4 + 2x^2 + 9 = (x^2 + 2x + 3)(x^2 - 2x + 3)$.

$$\therefore \frac{4x^2+8x}{x^4+2x^2+9} = \frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)}$$

Suppose

$$\frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)} = \frac{Ax+B}{x^2+2x+3} + \frac{Cx+D}{x^2-2x+3}$$

$\Rightarrow \quad 4x^2 + 8x = (Ax + B)(x^2 - 2x + 3) + (Cx + D)(x^2 + 2x + 3)$

$\Rightarrow \quad 4x^2 + 8x = (A + C)x^3 + (-2A + B + 2C + D)x^2$

$\quad + (3A - 2B + 3C + 2D)x + 3B + 3D \quad$ (I)

which is an identity in x.

Equating the coefficients of x^3, x^2, x, x^0 in I, we have

$0 = A + C \quad$ (i)

$4 = -2A + B + 2C + D \quad$ (ii)

$8 = 3A - 2B + 3C + 2D \quad$ (iii)

$0 = 3B + 2D \quad$ (iv)

Solving (i), (ii), (iii) and (iv), we get

$\boxed{A = 1}$, $\boxed{B = 2}$, $\boxed{C = -1}$ and $\boxed{D = -2}$

Hence the partial fractions are: $\dfrac{x+2}{x^2+2x+3} + \dfrac{-x-2}{x^2-2x+3}$

Exercise 5.3

Resolve the following into Partial Fractions:

1. $\dfrac{9x-7}{(x^2+1)(x+3)}$ **2.** $\dfrac{1}{(x^2+1)(x+1)}$ **3.** $\dfrac{3x+7}{(x^2+4)(x+3)}$

4. $\dfrac{x^2+15}{(x^2+2x+5)(x-1)}$ **5.** $\dfrac{x^2}{(x^2+4)(x+2)}$ **6.** $\dfrac{x^2+1}{x^3+1}$

7. $\dfrac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$ **8.** $\dfrac{1}{(x-1)^2(x^2+2)}$ **9.** $\dfrac{x^4}{1-x^4}$

10. $\dfrac{x^2-2x+3}{x^4+x^2+1}$

Case IV: when $Q(x)$ has repeated irreducible quadratic factors

If the polynomial $Q(x)$ contains a repeated irreducible quadratic factors $(a_n x^2 + bx + c)^n$, $n \geq 2$ and n is a +ve integer, then $\dfrac{P(x)}{Q(x)}$ may be written as the following identity:

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{a_1x^2 + bx + c} + \frac{A_2x + B_2}{(a_2x^2 + bx + c)^2} + \cdots\cdots + \frac{A_nx + B_n}{(a_nx^2 + bx + c)^n}$$

where $A_1, B_1, A_2, B_2, \ldots\ldots A_n, B_n$ are numbers to be found. The method is explained through the following example:

Example 1: Resolve $\frac{4x^2}{(x^2+1)^2(x-1)}$ into partial fractions.

Solution: Let $\frac{4x^2}{(x^2+1)^2(x-1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-1}$

$\Rightarrow \quad 4x^2 = (Ax + B)(x^2 + 1)(x - 1) + (Cx + D)(x - 1) + E(x^2 + 1)^2 \quad$ (i)

$\Rightarrow \quad 4x^2 = (A + E)x^4 + (-A + B)x^3 + (A - B + C + 2E)x^2$
$+(-A + B - C + D)x + (-B - D + E) \quad$ (ii)

Putting $x - 1 = 0 \Rightarrow x = 1$ in (i), we get

$4 = E(1 + 1)^2 \qquad \Rightarrow \boxed{E = 1}$

Equating the coefficients of x^4, x^3, x^2, x, in (ii), we get

$0 = A + E \quad \Rightarrow \quad A = -E \qquad \Rightarrow \boxed{A = -1}$

$0 = -A + B \quad \Rightarrow \quad B = A \qquad \Rightarrow \boxed{B = -1}$

$4 = A - B + C + 2E$

$\Rightarrow \quad C = 4 - A + B - 2E = 4 + 1 - 1 - 2 \qquad \Rightarrow \boxed{C = 2}$

$0 = -A + B - C + D$

$\Rightarrow \quad D = A - B + C = -1 + 1 + 2 = 2 \qquad \Rightarrow \boxed{D = 2}$

Hence partial fractions are: $\frac{-x-1}{x^2+1} + \frac{2x+2}{(x^2+1)^2} + \frac{1}{x-1}$

Exercise 5.4

Resolve into Partial Fractions.

1. $\frac{x^3+2x+2}{(x^2+x+1)^2}$ **2.** $\frac{x^2}{(x^2+1)^2(x-1)}$

3. $\frac{2x-5}{(x^2+2)^2(x-2)}$ **4.** $\frac{8x^2}{(x^2+1)^2(1-x^2)}$

5. $\frac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2}$ **6.** $\frac{2x^4-3x^3-4x}{(x^2+2)^2(x+1)^2}$

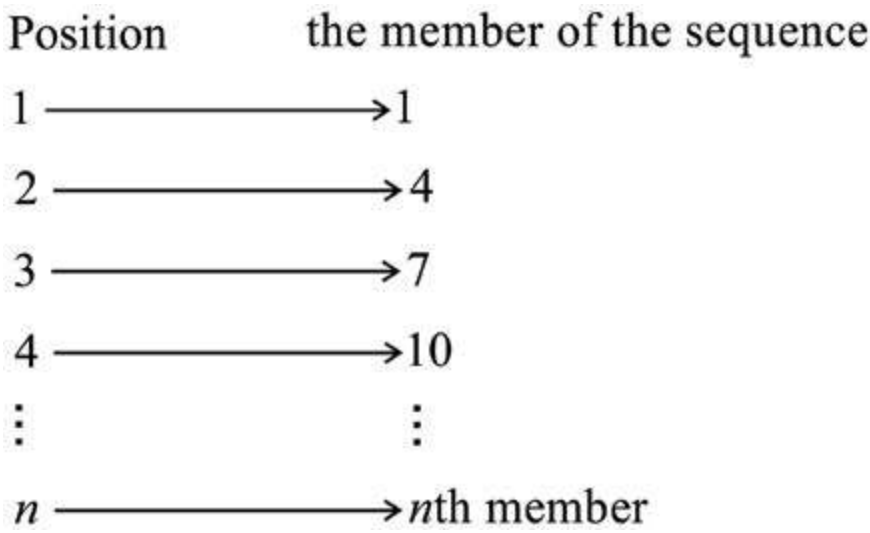
CHAPTER

6

Sequences and Series

6.1 Introduction

Sequences also called Progressions, are used to represent ordered lists of numbers. As the members of a sequence are in a definite order, so a correspondence can be established by matching them one by one with the numbers 1, 2, 3, 4,..... For example, if the sequence is 1, 4, 7, 10, ..., *n*th member, then such a correspondence can be set up as shown in the diagram below:

Position	the member of the sequence
1 ⟶	1
2 ⟶	4
3 ⟶	7
4 ⟶	10
⋮	⋮
n ⟶	nth member

Thus a sequence is a function whose domain is a subset of the set of natural numbers. A **sequence** is a special type of a function from a subset of N to R or C. Sometimes, the domain of a sequence is taken to be a subset of the set {0, 1, 2, 3,...}, i.e., the set of non-negative integers. If all members of a sequence are real numbers, then it is called a **real sequence**.

Sequences are usually named with letters *a, b, c* etc., and n is used instead of x as a variable. If a natural number n belongs to the domain of a sequence a, the corresponding element in its range is denoted by a_n. For convenience, a special notation a_n is adopted for $a(n)$and the symbol $\{a_n\}$ or $a_1, a_2, a_3,....,a_n,...$is used to represent the sequence a. The elements in the range of the sequence $\{a_n\}$ are called its terms; that is, a_1 is the first term, a_2 the second term and a_n the nth term or the general term.

For example, the terms of the sequence $\{n + (-1)^n\}$ can be written by assigning to n, the values 1, 2, 3 ,... If we denote the sequence by $\{b_n\}$, then

$b_n = n + (-1)^n$ and we have

$b_1 = 1 + (-1)^1 = 1 - 1 = 0$

$b_2 = 2 + (-1)^2 = 2 + 1 = 3$

$b_3 = 3 + (-1)^3 = 3 - 1 = 2$

$b_4 = 4 + (-1)^4 = 4 + 1 = 5$ etc.

If the domain of a sequence is a finite set, then the sequence is called a finite sequence otherwise, an infinite sequence.

Note: An infinite sequence has no last term.

Some examples of sequences are;

i) 1, 4, 9,...,121 ii) 1, 3, 5, 7, 9,...,21 iii) 1, 2, 4,...

iv) 1, 3, 7, 15, 31,... v) 1, 6, 20, 56,... vi) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9},...$

The sequences (i) and (ii) are finite whereas the sequences (iii) to (vi) are infinite.

6.2 Types of sequences

If we are able to find a pattern from the given initial terms of a sequence, then we can deduce a rule or formula for the terms of the sequence:

we can find any term of the given sequence giving corresponding value to n in the nth / general term a_n of a sequence.

Example 1: Write first two, 21st and 26th terms of the sequence whose general term is $(-1)^{n+1}$.

Solution: Given that $a_n = (-1)^{n+1}$. For getting required terms, we put n = 1, 2, 21 and 26.

$a_1 = (-1)^{1+1} = 1$

$a_2 = (-1)^{2+1} = -1$

$a_{21} = (-1)^{21+1} = 1$

$a_{26} = (-1)^{26+1} = -1$

Example 2: Find the sequence if $a_n - a_{n-1} = n + 1$ and $a_4 = 14$

Solution: Putting $n = 2, 3, 4$ in

$a_n - a_{n-1} = n + 1$, we have

$a_2 - a_1 = 3$ (i)

$a_3 - a_2 = 4$ (ii)

$a_4 - a_3 = 5$ (iii)

From (iii), $a_3 = a_4 - 5$

$= 14 - 5 = 9$ $(\because a_4 = 14)$

From (ii), $a_2 = a_3 - 4$

$= 9 - 4 = 5$ $(\because a_3 = 9)$

And from (i), $a_1 = a_2 - 3$

$= 5 - 3 = 2$

Thus the sequence is 2, 5, 9, 14, 20,...

Note: $a_5 - a_4 = 6 \Rightarrow a_5 = a_4 + 6 = 14 + 6 = 20$

Exercise 6.1

1. Write the first four terms of the following sequences, if

i) $a_n = 2n - 3$ ii) $a_n = (-1)^n n^2$ iii) $a_n = (-1)^n(2n - 3)$

iv) $a_n = 3n - 5$ v) $a_n = \dfrac{n}{2n+1}$ vi) $a_n = \dfrac{1}{2^n}$

vii) $a_n - a_{n-1} = n + 2, a_1 = 2$ viii) $a_n = na_{n-1}, a_1 = 1$

ix) $a_n = (n + 1)a_{n-1}, a_1 = 1$ x) $a_n = \dfrac{1}{a + (n-1)d}$

2. Find the indicated terms of the following sequences;

i) $2,6,11,17,\ldots a_7$ ii) $1,3,12,60,\ldots a_6$ iii) $1,\frac{3}{2},\frac{5}{4},\frac{7}{8},\ldots a_7$

iv) $1,1,-3,5,-7,9,\ldots a_8$ v) $1,-3,5,-7,9,-11,\ldots a_8$

3. Find the next two terms of the following sequences;

i) 7,9,12,16,... ii) 1,3,7,15,31,...

iii) –1,2,12,40,... iv) 1,–3,5,–7,9,–11...

6.3 Arithmetic Progression (A.P)

A sequence $\{a_n\}$ is an Arithmetic Sequence or Arithmetic progression (A.P), if $a_n - a_{n-1}$ is the same number for all $n \in N$ and $n > 1$. The difference $a_n - a_{n-1}$ $(n > 1)$ i.e., the difference of two consecutive terms of an A.P., is called the **common difference** and is usually denoted by ***d***.

Rule for the *n*th term of an A.P.:

We know that $a_n - a_{n-1} = d$ $(n > 1)$,

which implies $a_n = a_{n-1} + d$ $(n > 1)$...... (i)

Putting $n = 2, 3, 4,$...in (i) we get

$a_2 = a_1 + d = a_1 + (2 - 1)d$

$a_3 = a_2 + d = (a_1 + d) + d$

$= a_1 + 2d = a_1 + (3 - 1)d$

$a_4 = a_3 + d = (a_1 + 2d) + d$

$= a_1 + 3d = a_1 + (4 - 1)d$

Thus we conclude that

$$a_n = a_1 + (n - 1)d$$

where a_1 is the first term of the sequence.

We have observed that

$a_1 = a_1 + 0d = a_1 + (1 - 1)d$

$a_2 = a_1 + d = a_1 + (2 - 1)d$

$a_3 = a_2 + d = a_1 + (3 - 1)d$

$a_4 = a_3 + d = a_1 + (4 - 1)d$

Thus $a_1, a_1 + d, a_1 + 2d, \ldots, a_1 + (n - 1)d + \ldots$ is a general arithmetic sequence, with a_1, d as the first term and common difference respectively.

Note: $a_n = a_1 + (n - 1)d$ is called the nth term or general term of the A.P.

Example 1: Find the general term and the eleventh term of the A.P. whose first term and the common difference are 2 and –3 respectively. Also write its first four terms.

Solution: Here, $a_1 = 2, d = -3$

We know that $a_n = a_1 + (n-1)d$,

so $a_n = 2 + (n-1)(-3) = 2 - 3n + 3$

or $a_n = 5 - 3n$ (i)

Thus the general term of the A.P. is $5 - 3n$.

Putting $n = 11$ in (i), we have

$a_{11} = 5 - 3(11)$

$= 5 - 33 = -28$

We can find a_2, a_3, a_4 by putting $n = 2, 3, 4$ in (i), that is,

$a_2 = 5 - 3(2) = -1$

$a_3 = 5 - 3(3) = -4$

$a_4 = 5 - 3(4) = -7$

Hence the first four terms of the sequence are: 2, –1, –4, –7.

Example 2: If the 5th term of an A.P. is 13 and 17th term is 49, find a_n and a_{13}.

Solution: Given $a_5 = 13$ and $a_{17} = 49$.

Putting $n = 5$ in $a_n = a_1 + (n-1)d$, we have

$a_5 = a_1 + (5-1)d,$

$a_5 = a_1 + 4d$

or $13 = a_1 + 4d$ (i)

Also $a_{17} = a_1 + (17-1)d$

or $49 = a_1 + 16d$

or $49 = (a_1 + 4d) + 12d$

or $49 = 13 + 12d$ (by (i))

$\Rightarrow 12d = 36 \Rightarrow d = 3$

From (i), $a_1 = 13 - 4d = 13 - 4(3) = 1$

Thus $a_{13} = 1 + (13-1)3 = 37$ and

$a_n = 1 + (n-1)3 = 3n - 2$

Example 3: Find the number of terms in the A.P. if; $a_1 = 3$, $d = 7$ and $a_n = 59$.

Solution: Using $a_n = a_1 + (n-1)d$, we have

$59 = 3 + (n-1) \times 7$ $(\because a_n = 59, a_1 = 3 \text{ and } d = 7)$

or $56 = (n-1) \times 7 \Rightarrow (n-1) = 8 \Rightarrow n = 9$

Thus the terms in the A.P. are 9.

Example 4: If $a_{n-2} = 3n - 11$, find the nth term of the sequence.

Solution: Putting $n = 3, 4, 5$ in $a_{n-2} = 3n - 11$, we have

$a_1 = 3 \times 3 - 11 = -2$

$a_2 = 3 \times 4 - 11 = 1$

$a_3 = 3 \times 5 - 11 = 4$

Thus $a_n = a_1 + (n-1)d = -2 + (n-1) \times 3$ $(\because a_1 = -2, \text{ and } d = 3)$

$= 3n - 5$

Exercise 6.2

1. Write the first four terms of the following arithmetic sequences, if
   i) $a_1 = 5$ and other three consecutive terms are 23, 26, 29
   ii) $a_5 = 17$ and $a_9 = 37$ iii) $3a_7 = 7a_4$ and $a_{10} = 33$
2. If $a_{n-3} = 2n - 5$, find the nth term of the sequence.
3. If the 5th term of an A.P. is 16 and the 20th term is 46, what is its 12th term?
4. Find the 13th term of the sequence $x, 1, 2 - x, 3 - 2x, ...$
5. Find the 18th term of the A.P. if its 6th term is 19 and the 9th term is 31.
6. Which term of the A.P. 5, 2, –1,... is –85?
7. Which term of the A.P. –2, 4, 10,...is 148?
8. How many terms are there in the A.P. in which $a_1 = 11$, $a_n = 68$, $d = 3$?
9. If the nth term of the A.P. is $3n - 1$, find the A.P.
10. Determine whether (i) –19, (ii) 2 are the terms of the A.P. 17, 13, 9, ... or not.
11. If l, m, n are the pth, qth and rth terms of an A.P., show that
    i) $l(q - r) + m(r - p) + n(p - q) = 0$
    ii) $p(m - n) + q(n - l) + r(l - m) = 0$

12. Find the *n*th term of the sequence,

$$\left(\frac{4}{3}\right)^2, \left(\frac{7}{3}\right)^2, \left(\frac{10}{3}\right)^2, \ldots$$

13. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P., show that $b = \frac{2ac}{a+c}$.

14. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P, show that the common difference is $\frac{a-c}{2ac}$.

6.4 Arithmetic Mean (A.M)

A number A is said to be the A.M. between the two numbers a and b if a, A, b are in A.P. If d is the common difference of this A.P., then $A - a = d$ and $b - A = d$.

Thus $A - a = b - A$

or $2A = a + b$

$$\Rightarrow \quad A = \frac{a+b}{2}$$

Note: Middle term of three consecutive terms in A.P. is the A.M. between the extreme terms.

In general ,we can say that a_n is the A.M. between a_{n-1} and a_{n+1}, i.e.,

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

Example 1: Find three A.Ms between $\sqrt{2}$ and $3\sqrt{2}$.

Solution: Let A_1, A_2, A_3 be three A.Ms between $\sqrt{2}$ and $3\sqrt{2}$. Then

$\sqrt{2}, A_1, A_2, A_3, 3\sqrt{2}$ are in A.P.

Here $a_1 = \sqrt{2}, a_5 = 3\sqrt{2}$

Using $a_n = a_1 + (n-1)d$, we get

$a_5 = a_1 + (5-1)d$

or $3\sqrt{2} = \sqrt{2} + 4d$

$\Rightarrow \quad 3\sqrt{2} - \sqrt{2} = 4d$

$\Rightarrow \quad d = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Now $A_1 = a_1 + d = \sqrt{2} + \frac{1}{\sqrt{2}} = \frac{2+1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$,

$A_2 = A_1 + d = \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

$A_3 = A_2 + d = 2\sqrt{2} + \frac{1}{\sqrt{2}} = \frac{4+1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$

Therefore, $\frac{3}{\sqrt{2}}, 2\sqrt{2}, \frac{5}{\sqrt{2}}$ are three A.Ms between $\sqrt{2}$ and $3\sqrt{2}$.

6.4.1 *n* Arithmetic Means Between two given numbers

The n numbers $A_1, A_2, A_3, \ldots, A_n$ are called n arithmetic means between a and b if $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P.

Example 2: Find n A.Ms between a and b.

Solution: Let $A_1, A_2, A_3, \ldots, A_n$ be n arithmetic means between a and b.

Then $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P. in which $a_1 = a$ and $a_{n+2} = b$, so

$b = a + ((n+2)-1)d$ (where d is the common difference of the A.P.)

$= a + (n+1)d$

$\Rightarrow \quad d = \frac{b-a}{n+1}$

Thus $A_1 = a + d = a + \frac{b-a}{n+1} = \frac{na+b}{n+1}$

$A_2 = a + 2d = a + 2\left(\frac{b-a}{n+1}\right) = \frac{(n-1)a+2b}{n+1}$

$A_3 = a + 3d = a + 3\left(\frac{b-a}{n+1}\right) = \frac{(n-2)a+3b}{n+1}$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$A_n = a + nd = a + n\left(\frac{b-a}{n+1}\right) = \frac{a+nb}{n+1}$$

Exercise 6.3

1. Find A.M. between

i) $3\sqrt{5}$ and $5\sqrt{5}$ ii) $x-3$ and $x+5$

iii) $1-x+x^2$ and $1+x+x^2$

2. If 5, 8 are two A.Ms between a and b, find a and b.

3. Find 6 A.Ms. between 2 and 5.

4. Find four A.Ms. between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$.

5. Insert 7 A.Ms. between 4 and 8.

6. Find three A.Ms between 3 and 11.

7. Find n so that $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ may be the A.M. between a and b.

8. Show that the sum of n A.Ms. between a and b is equal to n times their A.M.

6.5 Series

The sum of an indicated number of terms in a sequence is called a series. For example, the sum of the first seven terms of the sequence $\{n^2\}$ is the series,

1 + 4 + 9 + 16 + 25 + 36 + 49.

The above series is also named as the 7th partial sum of the sequence $\{n^2\}$. If the number of terms in a series is finite, then the series is called a finite series, while a series consisting of an unlimited number of terms is termed as an infinite series.

Sum of first *n* terms of an arithmetic series:

For any sequence $\{a_n\}$, we have,

$S_n = a_1 + a_2 + a_3 + ... + a_n$

If $\{a_n\}$ is an A.P., then S_n can be written with usual notations as:

$$S_n = a_1 + (a_1+d) + (a_1+2d) + ... + (a_n-2d) + (a_n-d) + a_n \qquad \text{(i)}$$

If we write the terms of the series in the reverse order, the sum of n terms remains the same, that is,

$$S_n = a_n + (a_n-d) + (a_n-2d) + ... + (a_1+2d) + (a_1+d) + a_1 \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2S_n = (a_1+a_n) + (a_1+a_n) + (a_1+a_n) + ... + (a_1+a_n) + (a_1+a_n) + (a_1+a_n)$$
$$= (a_1+a_n) + (a_1+a_n) + (a_1+a_n) + ... \text{to } n \text{ term}$$
$$= n(a_1+a_n)$$

Thus $$S_n = \frac{n}{2}(a_1+a_n) \qquad \text{(iii)}$$

$$= \frac{n}{2}[a_1 + a_1 + (n-1)d]$$

or $$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

Example 1: Find the 19th term and the partial sum of 19 terms of the arithmetic series:

$$2 + \frac{7}{2} + 5 + \frac{13}{2} + ...$$

Solution: Here $a_1 = 2$ and $d = a_2 - a_1 = \frac{3}{2}$

Using $a_n = a_1 + (n-1)d$, we have,

$$a_{19} = 2 + (19-1)\frac{3}{2}$$
$$= 2 + 18\left(\frac{3}{2}\right) = 2 + 27 = 29$$

Using $S_n = \frac{n}{2}(a_1 + a_n)$, we have,

$$S_{19} = \frac{19}{2}(2+29) = \frac{19}{2}(31) = \frac{589}{2}$$

Example 2: Find the arithmetic series if its fifth term is 19 and $S_4 = a_9 + 1$.

Solution: Given that $a_5 = 19$, that is,

$$a_1 + 4d = 19 \quad \text{(i)}$$

Using the other given condition, we have,

$$S_4 = \frac{4}{2}[2a_1 + (4-1)d] = a_9 + 1$$

or $4a_1 + 6d = a_1 + 8d + 1$

$$3a_1 - 1 = 2d \quad \text{(ii)}$$

Substitution $2d = 3a_1 - 1$ in (i), gives

$$a_1 + 2(3a_1 - 1) = 19$$

or $7a_1 = 21 \Rightarrow a_1 = 3$

From (i), we have,

$$4d = 19 - a_1 = 19 - 3 = 16$$

$\Rightarrow \quad d = 4$

Thus the series is 3 + 7 +11 +15 +19 + ...

Example 3: How many terms of the series –9 –6 – 3 + 0 + ... amount to 66?

Solution: Here $a_1 = -9$ and $d = 3$ as $-6-(-9) = 3$ and $-3-(-6) = 3$.

Let $S_n = 66$

Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$, we have,

$$66 = \frac{n}{2}[2(-9) + (n-1)3]$$

or $132 = n[3n - 21] \Rightarrow 44 = n(n-7)$

or $n^2 - 7n - 44 = 0 \Rightarrow n = \frac{7 \pm \sqrt{49+176}}{2} = \frac{7 \pm \sqrt{225}}{2}$

$$= \frac{7 \pm 15}{2} \Rightarrow n = 11, -4$$

But *n* cannot be negative in this case, so *n* = 11, that is, the sum of eleven terms amount to 66.

Exercise 6.4

1. Find the sum of all the integral multiples of 3 between 4 and 97.

2. Sum the series

i) $-3 + (-1) + 1 + 3 + 5 + ... + a_{16}$

ii) $\frac{3}{\sqrt{2}} + 2\sqrt{2} + \frac{5}{\sqrt{2}} + ... + a_{13}$

iii) $1.11 + 1.41 + 1.71 + ... + a_{10}$.

iv) $-8 - 3\frac{1}{2} + 1 + ... + a_{11}$

v) $(x-a) + (x+a) + (x+3a) + ...$ to n terms.

vi) $\frac{1}{1-\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1+\sqrt{x}} + ...$ to n terms.

vii) $\frac{1}{1+\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1-\sqrt{x}} + ...$ to n terms.

3. How many terms of the series

i) $-7 + (-5) + (-3) + ...$ amount to 65?

ii) $-7 + (-4) + (-1) + ...$ amount to 114?

4. Sum the series

i) $3 + 5 - 7 + 9 + 11 - 13 + 15 + 17 - 19 + ...$ to $3n$ terms.

ii) $1 + 4 - 7 + 10 + 13 - 16 + 19 + 22 - 25 + ...$ to $3n$ terms.

5. Find the sum of 20 terms of the series whose *r*th term is $3r + 1$.

6. If $S_n = n(2n-1)$, then find the series.
7. The ratio of the sums of n terms of two series in A.P. is $3n+2 : n+1$. Find the ratio of their 8th terms.
8. If S_2, S_3, S_5 are the sums of $2n, 3n, 5n$ terms of an A.P., show that $S_5 = 5(S_3 - S_2)$.
9. Obtain the sum of all integers in the first 1000 integers which are neither divisible by 5 nor by 2.
10. S_8 and S_9 are the sums of the first eight and nine terms of an A.P., find S_9 if $50S_9 = 63S_8$ and $a_1 = 2$ (Hint $:= S_9 - S_8 = a_9$)
11. The sum of 9 terms of an A.P. is 171 and its eighth term is 31. Find the series.
12. The sum of S_9 and S_7 is 203 and $S_9 - S_7 = 49$, S_7 and S_9 being the sums of the first 7 and 9 terms of an A.P. respectively. Determine the series.
13. S_7 and S_9 are the sums of the first 7 and 9 terms of an A.P. respectively. If $\dfrac{S_9}{S_7} = \dfrac{18}{11}$ and $a_7 = 20$, find the series.
14. The sum of three numbers in an A.P. is 24 and their product is 440. Find the numbers.
15. Find four numbers in A.P. whose sum is 32 and the sum of whose squares is 276.
16. Find the five numbers in A.P. whose sum is 25 and the sum of whose squares is 135.
17. The sum of the 6th and 8th terms of an A.P. is 40 and the product of 4th and 7th term is 220. Find the A.P.
18. If a^2, b^2 and c^2 are in A.P., show that $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P.

6.6 Word Problems on A.p.

Example 1: Tickets for a certain show were printed bearing numbers from 1 to 100. Odd number tickets were sold by receiving paisas equal to thrice of the number on the ticket while even number tickets were issued by receiving paisas equal to twice of the number on the ticket. How much amount was received by the issuing agency?

Solution: Let S_1 and S_2 be the amounts received for odd number and even number tickets

respectively. Then

$$S_1 = 3[1+3+5+....+99] \text{ and } S_2 = 2[2+4+6+...+100]$$

Thus $S_1 + S_2 = 3\times\dfrac{50}{2}(1+99) + 2\times\dfrac{50}{2}(2+100)$, [$\because$ There are 50 terms in each series]

= 7500 + 5100 = 12600

Hence the total amount received by the issuing agency = 12600 paisas = Rs.126

Example 2: A man repays his loan of Rs.1120 by paying Rs.15 in the first installment and then increases the payment by Rs.10 every month. How long will it take to clear his loan?

Solution: It is given that the first installment (in Rs.) is 15 and the monthly increase in payment (in Rs.) is 10.

Here $a_1 = 15$ and $d = 10$

Let the time required (in months) to clear his loan be n. Then

$S_n = 1120$, that is,

$$1120 = \frac{n}{2}[2\times 15 + (n-1)10] = \frac{n}{2}[30 + (n-1)10]$$

$$= \frac{n}{2}\times 10[3 + (n-1)] = 5n(n+2)$$

or $224 = n(n+2) \Rightarrow n^2 + 2n - 224 = 0$

$$\Rightarrow \quad n = \frac{-2\pm\sqrt{4+896}}{2} = \frac{-2\pm\sqrt{900}}{2}$$

$$= \frac{-2\pm 30}{2}$$

$$= 14, -16$$

But n can not be negative, so n = 14, that is, the time required to clear his loan is 14 months.

Example 3: A manufacturer of radio sets produced 625 units in the 4th year and 700 units in the 7th year. Assuming that production uniformly increases by a fixed number every year,

find

i) The production in the first year
ii) The total production in 8 years
iii) The production in the 11th year.

Solution: Let a_1 be the number of units produced in the first year and d be the uniform increase in production every year. Then the sequence of products in the successive years is

$$a_1, a_1 + d, a_1 + 2d, \ldots$$

By the given conditions, we have

$a_4 = 625$ and a_7 700, that is,

$$a_1 + 3d = 625 \qquad \text{(I)}$$

and $$a_1 + 6d = 700 \qquad \text{(II)}$$

Subtracting (I) from (II), we get

$$3d = 75 \Rightarrow d = 25$$

i) From (I), $a_1 + 3(25) = 625 \Rightarrow a_1 = 625 - 75 = 550$

Thus the production in the first year is 550 units.

ii) $$S_8 = \frac{8}{2}[2 \times 550 + (8-1)25]$$
$$= 4[1100 + 175] = 4[1275] = 5100$$

Thus the production in 8 years is 5100 units.

iii) $$a_{11} = a_1 + (11-1)d$$
$$= 550 + 10 \times 25 = 550 + 250 = 800$$

Thus the production in the 11th year is 800 units.

Exercise 6.5

1. A man deposits in a bank Rs. 10 in the first month; Rs. 15 in the second month; Rs. 20 in the third month and so on. Find how much he will have deposited in the bank by the 9th month.
2. 378 trees are planted in rows in the shape of an isosceles triangle, the numbers in successive rows decreasing by one from the base to the top. How many trees are there in the row which forms the base of the triangle?
3. A man borrows Rs. 1100 and agree to repay with a total interest of Rs. 230 in 14 installments, each installment being less than the preceding by Rs. 10. What should be his first installment?
4. A clock strikes once when its hour hand is at one, twice when it is at two and so on. How many times does the clock strike in twelve hours ?
5. A student saves Rs.12 at the end of the first week and goes on increasing his saving Rs.4 weekly. After how many weeks will he be able to save Rs.2100?
6. An object falling from rest, falls 9 meters during the first second, 27 meters during the next second, 45 meters during the third second and so on.
   i) How far will it fall during the fifth second?
   ii) How far will it fall up to the fifth second?
7. An investor earned Rs.6000 for year 1980 and Rs. 12000 for year 1990 on the same investment. If his earning have increased by the same amount each year, how much income he has received from the investment over the past eleven years?
8. The sum of interior angles of polygon having sides 3,4,5,...etc. form an A.P. Find the sum of the interior angles for a 16 sided polygon.
9. The prize money Rs. 60,000 will be distributed among the eight teams according to their positions determined in the match-series. The award increases by the same amount for each higher position. If the last place team is given Rs.4000, how much will be awarded to the first place team?
10. An equilateral triangular base is filled by placing eight balls in the first row, 7 balls in the second row and so on with one ball in the last row. After this base layer, second layer is formed by placing 7 balls in its first row, 6 balls in its second row and so on with one ball in its last row. Continuing this process, a pyramid of balls is formed with one ball on top. How many balls are there in the pyramid?

6.7 Geometric Progression (G.P)

A sequence $\{a_n\}$ is a geometric sequence or geometric progression if $\frac{a_n}{a_{n-1}}$ is the same non-zero number for all $n \in N$ and $n > 1$. The quotient $\frac{a_n}{a_{n-1}}$ is usually denoted by ***r*** and is called **common ratio** of the G.P .It is Clear that *r* is the ratio of any term of the G.P., to its predecessor. The common ratio $r = \frac{a_n}{a_{n-1}}$ is defined only if $a_{n-1} \neq 0$, i.e., no term of the geometric sequence is zero.

Rule for *n*th term of a G.P.: Each term after the first term is an *r* multiple of its preceding term. Thus we have,

$$a_2 = a_1 r = a_1 r^{2-1}$$
$$a_3 = a_2 r = (a_1 r)r = a_1 r^2 = a_1 r^{3-1}$$
$$a_4 = a_3 r = (a_1 r^2)r = a_1 r^3 = a_1 r^{4-1}$$
$$\vdots$$

$a_n = a_1 r^{n-1}$ which is the general term of a G.P.

Example 1: Find the 5th term of the G.P., 3,6,12,...

Solution: Here $a_1 = 3,\ a_2 = 6,\ a_3 = 12$, therefore, $r = \frac{a_2}{a_1} = \frac{6}{3} = 2$

Using $a_n = a_1 r^{n-1}$ for $n = 5$, we have,

$$a_5 = a_1 r^{5-1} = 3.2^{5-1} = 3.2^4 = 48$$

Example 2: Find a_n if $a_4 = \frac{8}{27}$ and $a_7 = \frac{-64}{729}$ of a G.P.

Solution: To find a_n we have to find a_1 and r.

Using $a_n = a_1 r^{n-1}$ (i)

$a_4 = a_1 r^{4-1} = a_1 r^3$, so $a_1 r^3 = \frac{8}{27}$ (ii)

And $a_7 = a_1 r^{7-1} = a_1 r^6$, so $a_1 r^6 = \frac{-64}{729}$ (iii)

Thus $\frac{a_7}{a_4} = \frac{-64/729}{8/27} = -\frac{8}{27}$ or $r^3 = \left(-\frac{2}{3}\right)^3$ $\left(\because \frac{a_7}{a_4} = \frac{a_1 r^6}{a_1 r^3} = r^3\right)$

$\Rightarrow r = -\frac{2}{3}$ (taking only real value of *r*)

Put $r^3 = -\frac{8}{27}$ in (ii), to obtain a_1 that is,

$$a_1\left(-\frac{8}{27}\right) = \frac{8}{27} \Rightarrow a_1 = -1$$

Now putting $a_1 = -1$ and $r = -\frac{2}{3}$ in (i), we get,

$$a_n = (-1)\left(-\frac{2}{3}\right)^{n-1} = (-1)(-1)^{n-1}.\left(\frac{2}{3}\right)^{n-1} = (-1)^n\left(\frac{2}{3}\right)^{n-1} \text{ for } n \geq 1$$

Example 3: If the numbers 1, 4 and 3 are added to there consecutive terms of G.P., the resulting numbers are in A.P. Find the numbers if their sum is 13.

Solution: Let a, ar, ar^2 be three consecutive numbers of the G.P. Then

$a + ar + ar^2 = 13 \Rightarrow a(1 + r + r^2) = 13$ (i)

and $a+1, ar+4, ar^2+3$ are in A.P., according to the given condition.

Thus $ar + 4 = \frac{(a+1) + (ar^2+3)}{2}$

$$\Rightarrow 2ar + 8 = ar^2 + a + 4$$
$$\Rightarrow a(r^2 - 2r + 1) = 4$$
$$\Rightarrow a(r^2 + r + 1) - 3ar = 4 \quad (\because r^2 - 2r + 1 = (r^2 + r + 1) - 3r)$$
$$\Rightarrow 13 - 3ar = 4 \quad (\because a(1 + r + r^2) = 13)$$

or $3ar = 13 - 4 \Rightarrow ar = 3$ (ii)

Using $a=\frac{3}{r}$, (i) becomes

$$\frac{3}{r}(1+r+r^2)=13$$

or $3r^2-10r+3=0$

$\Rightarrow$ $r=\frac{10\pm\sqrt{100-36}}{6}=\frac{10\pm 8}{6}$

$r=3$ or $r=\frac{1}{3}$

If $r=3$ then $a=1$ (using $ar=3$)

and if $r=\frac{1}{3}$ then $a=9$ (using $ar=3$)

Thus the numbers are 1,3,9 or 9,3,1

Exercise 6.6

1. Find the 5th term of the G.P.: 3,6,12,...
2. Find the 11th term of the sequence, $1+i, 2, \frac{4}{1+i}, \ldots$
3. Find the 12th term of $1+i, 2i, -2+2i, \ldots$
4. Find the 11th term of the sequence, $1+i, 2, 2(1-i)$
5. If an automobile depreciates in value 5% every year, at the end of 4 years what is the value of the automobile purchased for Rs.12,000?
6. Which term of the sequence:
7. If a, b, c, d are in G.P, prove that
   i) $a-b, b-c, c-d$ are in G.P.
   ii) $a^2-b^2, b^2-c^2, c^2-d^2$ are in G.P.
   iii) $a^2+b^2, b^2+c^2, c^2+d^2$ are in G.P.
8. Show that the reciprocals of the terms of the geometric sequence $a_1, a_1r^2, a_1r^4, \ldots$ form another geometric sequence.

9. Find the nth term of the geometric sequence if; $\frac{a_5}{a_3}=\frac{4}{9}$ and $a_2=\frac{4}{9}$
10. Find three, consecutive numbers in G.P whose sum is 26 and their product is 216.
11. If the sum of the four consecutive terms of a G.P is 80 and A.M of the second and the fourth of them is 30. Find the terms.
12. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in G.P. show that the common ratio is $\pm\sqrt{\frac{a}{c}}$
13. If the numbers 1,4 and 3 are subtracted from three consecutive terms of an A.P., the resulting numbers are in G.P. Find the numbers if their sum is 21.
14. If three consecutive numbers in A.P. are increased by 1, 4, 15 respectively, the resulting numbers are in G.P. Find the original numbers if their sum is 6.

6.8 Geometric Means

A number G is said to be a geometric mean (G.M.) between two numbers a and b if a, G, b are in G.P. Therefore,

$$\frac{G}{a}=\frac{b}{G} \Rightarrow G^2=ab \pm\Rightarrow G=\sqrt{ab}$$

6.8.1 n Geometric Means Between two given numbers

The n numbers $G_1, G_2, G_3, \ldots, G_n$ are called n geometric means between a and b if $a, G_1, G_2, G_3, \ldots, G_n, b$ are in G.P.

Thus we have, $b=ar^{(n+2)-1}$ where r is the common ratio,

or $ar^{n+1}=b$

$$\Rightarrow \quad r=\left(\frac{b}{a}\right)^{1/n+1}$$

Thus $G_1=ar=a\left(\frac{b}{a}\right)^{1/n+1}$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/n+1}$$

$$G_3 = ar^3 = a\left(\frac{b}{a}\right)^{3/n+1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{n/n+1}$$

Note: $G_1.G_2.G_3....G_n = a^n\left(\frac{b}{a}\right)^{\frac{(1+2+3....+n)}{n+1}} = a^n\left(\frac{b}{a}\right)^{\frac{n}{2}}$

and $\sqrt[n]{(G_1.G_2.G_3....G_n)} = a\left(\frac{b}{a}\right)^{1/2} = \sqrt{ab}$

= *G*, the geometric mean between *a* and *b*

Example 1: Find the geometric mean between 4 and 16.

Solution: Here *a* = 4, *b* = 16, therefore

$$G = \pm\sqrt{ab} = \pm\sqrt{4\times 16}$$
$$= \pm\sqrt{64} = \pm 8$$

Thus the geometric mean may be +8 or –8. Inserting each of two G.Ms. between 4 and 16, we have two geometric sequences 4, 8, 16 and 4, –8, 16. In the first case *r* = 2 and in the second case *r* = –2 .

Example 2: Insert three G.Ms. between 2 and $\frac{1}{2}$.

Solution: Let G_1, G_2, G_3 be three G.Ms between 2 and $\frac{1}{2}$. Therefore 2, $G_1, G_2, G_3, \frac{1}{2}$ are in G.P.

Here $a_1 = 2,\quad a_5 = \frac{1}{2}$ and $n = 5$

Using $a_n = a_1 r^{n-1}$ we have,

$$a_5 = a_1 r^{5-1} \quad \text{i.e.,} \quad a_5 = a_1 r^4 \qquad \text{(i)}$$

Now substituting the values of a_5 and a_1 in (i) we have

$$\frac{1}{2} = 2r^4 \quad \text{or} \quad r^4 = \frac{1}{4} \qquad \text{(ii)}$$

Taking square root of (ii), we get,

$$r^2 = \pm\frac{1}{2}$$

So, we have, $r^2 = \frac{1}{2}$ or $r^2 = -\frac{1}{2} = \frac{i^2}{2} \quad (\because\ -1 = i^2)$

$$\Rightarrow \quad r = \pm\frac{1}{\sqrt{2}} \quad \text{or} \quad r = \pm\frac{1}{\sqrt{2}}i$$

when $r = \frac{1}{\sqrt{2}}$, then $G_1 = 2\cdot\frac{1}{\sqrt{2}} = \sqrt{2}, G_2 = 2\left(\frac{1}{\sqrt{2}}\right)^2 = 1, G_3 = 2\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{\sqrt{2}}$

when $r = \frac{-1}{\sqrt{2}}$, then $G_1 = 2\left(\frac{-1}{\sqrt{2}}\right) = -\sqrt{2}, G_2 = 2\left(\frac{-1}{\sqrt{2}}\right)^2 = 1, G_3 = 2\left(\frac{-1}{\sqrt{2}}\right)^3 = -\frac{1}{\sqrt{2}}$

when $r = \frac{i}{\sqrt{2}}$, then $G_1 = 2\times\frac{i}{\sqrt{2}} = \sqrt{2}i, G_2 = 2\left(\frac{i}{\sqrt{2}}\right)^2 = -1, G_3 = 2\left(\frac{i}{\sqrt{2}}\right)^3 = -\frac{i}{\sqrt{2}}$

when $r = \frac{-i}{\sqrt{2}}$, then $G_1 = 2\left(\frac{-i}{\sqrt{2}}\right) = -\sqrt{2}\,i, G_2 = 2\left(\frac{-i}{\sqrt{2}}\right)^2 = -1, G_3 = 2\left(\frac{-i}{\sqrt{2}}\right)^3 = \frac{i}{\sqrt{2}}$

Note: The real values of *r* are usually taken but here other cases are considered to widen the out-look of the students.

Example 3: If *a*, *b*, *c* and *d* are in G.P. show that *a* + *b*, *b* + *c*, *c* + *d* are in G.P.

Solution: Since *a*, *b*, *c* are in G.P therefore,

$$ac = b^2 \qquad \text{(i)}$$

Also *b, c, d* are in G.P., so we have

$$bd = c^2 \qquad \text{(ii)}$$

Multiplying both sides, of (ii) by b, we get

$b^2 d = bc^2 \Rightarrow acd = bc^2 \qquad (\because ac = b^2)$

$\Rightarrow ad = bc \qquad$ (iii)

Now $ad + bc = bc + bc \qquad [\because ad = bc]$

i.e., $ad + bc = 2bc \qquad$ (iv)

Adding (i), (ii), and (iv), we have

$ac + bd + ad + bc = b^2 + c^2 + 2bc$

or $(a+b)c + (a+b)d = (b+c)^2$

or $(a+b)(c+d) = (b+c)^2$

$\Rightarrow$ $a+b, b+c, c+d$ are in G.P.

Exercise 6.7

1. Find G.M. between
   i) −2 and 8 ii) $-2i$ and $8i$
2. Insert two G.Ms. between
   i) 1 and 8 ii) 2 and 16
3. Insert three G.Ms. between
   i) 1 and 16 ii) 2 and 32
4. Insert four real geometric means between 3 and 96.
5. If both *x* and *y* are positive distinct real numbers, show that the geometric mean between *x* and *y* is less than their airthmetic mean.
6. For what value of *n*, $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the positive geometric mean between *a* and *b*?
7. The A.M. of two positive integral numbers exceeds their (positive) G.M. by 2 and their sum is 20, find the numbers.
8. The A.M. between two numbers is 5 and their (positive) G.M. is 4. Find the numbers.

6.9 Sum of n terms of a Geometric Series

For any sequence $\{a_n\}$, we have

$S_n = a_1 + a_2 + a_3 + ... + a_n$

If the sequence $\{a_n\}$ is a geometric sequence, then

$S_n = a_1 + a_1 r + a_1 r^2 + ... a_1 r^{n-1} \qquad$ (i)

Multiplying both sides of (i) by $1-r$ we get

$(1-r)S_n = (1-r)\{a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-1}\}$

$= (1-r)\{a_1(1 + r + r^2 + ... + r^{n-1})\}$

$= a_1\{(1-r)(1 + r + r^2 + ... + r^{n-1}\}$

$= a_1\{(1 + r + r + ... + r^{n-1}) - (r + r^2 + ... + r^n)\}$

$= a_1(1 - r^n)$

or $S_n = \frac{a_1(1-r^n)}{1-r} \qquad (r \neq 1)$

For convenience we use:

$S_n = \frac{a_1(1-r^n)}{1-r} \qquad \text{if } |r| < 1$

and $S_n = \frac{a_1(r^n - 1)}{r-1} \qquad \text{if } |r| > 1$

Example 1: Find the sum of *n* terms of the geometric series if $a_n = (-3)\left(\frac{2}{5}\right)^n$.

Solution: We can write $(-3)\left(\frac{2}{5}\right)^n$ as:

$-3\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^{n-1} = \left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$, that is,

$a_n = \left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$

Identifying $\left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$ with $a_1 r^{n-1}$,

we have, $a_1 = \frac{6}{5}$ and $r \ \frac{2}{5} \ 1$

Thus $S_n = \frac{a_1(1-r^n)}{1-r} = \frac{-\frac{6}{5}\left[1-\left(\frac{2}{5}\right)^n\right]}{1-\frac{2}{5}}$

$$= \left(-\frac{6}{5}\right)\left(\frac{5}{3}\right)\left[1-\left(\frac{2}{5}\right)^n\right] = (-2)\left[1 \quad \left(\frac{2}{5}\right)^n\right]$$

Example 2: The growth of a certain plant is 5% of its length monthly. When will the plant be of 4.41 cm if its initial length is 4 cm?

Solution: Let the initial length be l cm.Then at the end of one month, the plant will be of length $l + \left(1 \times \frac{5}{100}\right) = l + \frac{l}{20} = \frac{21}{20}l$.

The length of the plant at the end of second month $= \frac{21}{20}l + \frac{21}{20}l \times \frac{5}{100}$

$$= \frac{21}{20}l\left(1+\frac{1}{20}\right) = \left(\frac{21}{20}\right)^2 l$$

So, the sequence of lengths at the end of successive months is, $\frac{21}{20}l, \left(\frac{21}{20}\right)^2 l, \left(\frac{21}{20}\right)^3 l, ...$

Here $a_n = \left(\frac{21}{20}\right) l \times \left(\frac{21}{20}\right)^{n-1} = \left(\frac{21}{20}\right)^n l$ $\qquad \left(\because a_1 = \frac{21}{20}l, r = \frac{21}{20}\right)$

Thus $4.41 = \left(\frac{21}{20}\right)^n \times 4$ $\qquad$ ($\because$ initial length = 4 cm)

or $\left(\frac{21}{20}\right)^n = \frac{4.41}{4} = \frac{441}{400} = \left(\frac{21}{20}\right)^2$ which gives n = 2

6.10 The Infinite Geometric Series

Consider the series

$$a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-1} + ...,$$

then $\quad S_n = a_1 + a_1 r + ... + a_1 r^{n-1} = \frac{a_1(1-r^n)}{1-r} \qquad (r \neq 1)$

But we do not know how to add infinitely many terms of the series.

If $S_n \to a$ limit as $n \to \infty$ then the series is said to be convergent.

If S_n increases indefinitely as n becomes very large then we say that S_n does not exist and the series is said to be divergent.

Case I: If $|r| < 1$,

then r^n can be made as small as we like by taking n sufficiently large, that is,

$r^n \to 0 \quad$ as $\quad n \to \infty$

Obviously $S_n \to \frac{a_1}{1-r}$ when $n \to \infty$

In other words we can say that the series converges to the sum $\frac{a_1}{1-r}$ that is,

$$S = \lim_{n\to\infty} S_n = \frac{a_1}{1-r}$$

Case II: If $|r| > 1$,

then r^n does not tend to zero when $n \to \infty$

i.e., S_n does not tend to a limit and the series does not converge in this case so the series is divergent.

For example, if we take $a_1 = 1, r = 2$,

then the series, will be

1 + 2 + 4 + 8 + ...

and we have $S_1 = 1, = S_2 = 3, S_3 = 7, S_4 = 15, ..., S_n = 2^n - 1$, i.e., $S_1, S_2, S_3, ..., S_n$ is a sequence of ever increasing numbers.

In other words we can say that S_n increases indefinitely as $n \to \infty$. Thus the series does not converge.

Case III**:** If $r = 1$, then the series becomes

$$a_1 + a_1 + a_1 + a_1 + \ldots$$

and $S_n = na_1$. In this case S_n does not tend to a limit when $n \to \infty$ and the series does not converge.

Case IV**:** If $r = -1$, then the series becomes

$$a_1 - a_1 + a_1 - a_1 + a_1 - a_1 + \ldots$$

and $S_n = \dfrac{a_1 - (-1)^n a_1}{2}$

i.e., $S_n = a_1$ if *n* is positive odd integer.

$S_n = 0$ if *n* is positive even integer.

Thus S_n does not tend to a definite number when $n \to \infty$. In such a case we say that the series is oscillatory.

Example 3: Find the sum of the infinite G.P. $2, \sqrt{2}, 1, \ldots$

Solution: Here $a_1 = 2$

$r = \dfrac{a_2}{a_1} = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$ and

$$S = \frac{2}{1 - \frac{1}{\sqrt{2}}} \qquad \left(\because \frac{1}{\sqrt{2}} \quad 1\right)$$

$$= \frac{2\sqrt{2}}{\sqrt{2} - 1} = \frac{2\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} \quad \frac{4 + 2\sqrt{2}}{2 - 1} \quad 4 \quad 2\sqrt{2}$$

Example 4: Convert the recurring decimal $2.\dot{2}\dot{3}$ into an equivalent common fraction (vulgur fraction).

Solution: $2.\dot{2}\dot{3}$ = 2.232323 ....

= 2 + {.23 + .0023 + .000023 +...}

$$= 2\left[\frac{.23}{1 - \frac{1}{100}}\right] \qquad \left(\because \frac{.0023}{.23} \quad \frac{1}{100}\right)$$

$$= 2 + \frac{100 \times .23}{99} \quad 2\,\frac{23}{99}$$

$$= \frac{198 + 23}{99} = \frac{221}{99}$$

Example 5: The sum of an infinite geometric series is half the sum of the squares of its terms. If the sum of its first two terms is $\frac{9}{2}$, find the series.

Solution: Let the series be

$$a_1 + a_1 r + a_1 r^2 + \ldots \qquad \text{(i)}$$

Then the series whose terms are the squares of the terms of the above series is

$$a_1^2 + a_1^2 r^2 + a_1^2 r^4 + \ldots \qquad \text{(ii)}$$

Let S_1 and S_2 be the sum of the series (i) and (ii) respectively. Then

$$S_1 = \frac{a_1}{1 - r} \qquad \text{(iii)}$$

and $$S_2 = \frac{a_1^2}{1 - r^2} \qquad \text{(iv)}$$

By the first given condition, we have.

$$S_1 = \frac{1}{2} S_2 \Rightarrow \frac{a_1}{1 - r} = \frac{1}{2}\left(\frac{a_1^2}{1 - r^2}\right)$$

$$\Rightarrow a_1 = 2(1 + r) \qquad \text{(v)}$$

From the other given condition, we get

$$a_1 + a_1 r = \frac{9}{2} \Rightarrow a_1(1+r) = \frac{9}{2} \qquad \text{(vi)}$$

Substituting $a_1 = 2(1+r)$ in (vi), gives

$$2(1+r)\,(1+r) = \frac{9}{2} \Rightarrow (1+r)^2 = \frac{9}{4}$$

$$\Rightarrow 1 + r = \pm\frac{3}{2}$$

$$\Rightarrow r = \frac{1}{2}, -\frac{5}{2}$$

For $r = -\frac{5}{2}$, $|r| = \frac{5}{2} > 1$, so we cannot take $r = -\frac{5}{2}$.

if $r = \frac{1}{2}$, then $a_1 = 2(1+\frac{1}{2}) = 3$ $\quad [\because a_1 = 2(1+r)]$

Hence the series is $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots$

Example 6: If $a = 1 - x + x^2 - x^3 + \ldots \quad |x| < 1$

$b = 1 + x + x^2 + x^3 + \ldots \quad |x| < 1$

show that $2ab = a + b$.

Solution: $a = \frac{1}{1-(-x)} \qquad (\because r = -x)$

or $a = \frac{1}{1+x} \Rightarrow 1 + x = \frac{1}{a} \qquad \text{(i)}$

and $b = \frac{1}{1-x} \qquad (\because r = x)$

$\Rightarrow \quad 1 - x = \frac{1}{b} \qquad \text{(ii)}$

Adding (i) and (ii), we obtain

$2 = \frac{1}{a} + \frac{1}{b}$, which implies that

$$2ab = a + b$$

Exercise 6.8

1. Find the sum of first 15 terms of the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \ldots$

2. Sum to *n* terms, the series

i) .2 + .22 + .222 + ... ii) 3 + 33 + 333 + ...

3. Sum to *n* terms the series

i) $1 + (a+b) + (a^2 + ab + b^2) + (a^3 + a^2b + ab^2 + b^3) + \ldots$

ii) $r + (1+k)r^2 + (1+k+k^2)r^3 + \ldots$

4. Sum the series $2 + (1-i) + \left(\frac{1}{i}\right) + \ldots$ to 8 terms.

5. Find the sums of the following infinite geometric series:

i) $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots$ ii) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ iii) $\frac{9}{4} + \frac{3}{2} + 1 + \frac{2}{3} + \ldots$

iv) 2 + 1 + 0.5 + ... v) $4 + 2\sqrt{2} + 2 + \sqrt{2} + 1 + \ldots$

vi) 0.1 + 0.05 + 0.025 + ...

6. Find vulgar fractions equivalent to the following recurring decimals.

i) $1.\dot{3}\dot{4}$ ii) $0.\dot{7}$ iii) $0.\dot{2}\dot{5}\dot{9}$

iv) $1.\dot{5}\dot{3}$ v) $0.\dot{1}\dot{5}\dot{9}$ vi) $1.1\dot{4}\dot{7}$

7. Find the sum to infinity of the series; $r + (1+k)r^2 + (1+k+k^2)r^3 + \ldots$ *r* and *k* being proper fractions.

8. If $y = \frac{x}{2} + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \ldots$ and if $0 < x < 2$, then prove that $x = \frac{2y}{1+y}$

9. If $y = \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \ldots$ and if $0 < x < \frac{3}{2}$, then show that $x = \frac{3y}{2(1+y)}$

10. A ball is dropped from a height of 27 meters and it rebounds two-third of the distance it falls. If it continues to fall in the same way what distance will it travel before coming to rest?

11. What distance will a ball travel before coming to rest if it is dropped from a height of

75 meters and after each fall it rebounds $\frac{2}{5}$ of the distance it fell?

12. If $y = 1 + 2x + 4x^2 + 8x^3 + \ldots$

i) Show that $x = \frac{y-1}{2y}$

ii) Find the interval in which the series is convergent.

13. If $y = 1 + \frac{x}{2} + \frac{x^2}{4} + \ldots$

i) Show that $x = 2\left(\frac{y-1}{y}\right)$

ii) Find the interval in which the serieis is convergent.

14. The sum of an infinite geometric series is 9 and the sum of the squares of its terms is $\frac{81}{5}$. Find the series.

6.11 Word Problems on G.P.

Example 1: A man deposits in a bank Rs. 20 in the first year; Rs. 40 in the second year; Rs. 80 in the third year and so on. Find the amount he will have deposited in the bank by the seventh year.

Solution: The deposits in the succcessive yesrs are

20,40,80,... which is a geometric sequence with

$a_1 = 20$ and r 2

The sum of the seven terms of the above sequence is the total amount deposited in the bank upto the seventh year, so we have to find S_7, that is,

the required deposit in Rs. $= \frac{20(2^7-1)}{2-1} = \frac{20(2^7-1)}{1}$

$= 20(128 - 1) = 20 \times 127$

$= 2540$

Thus the amount deposited in the bank upto the seventh year is Rs. 2540.

Example 2: A person invests Rs.2000/- at 4% interest compounded annually. What the total amount will he get after 5 years?

Solution: Let the principal amtount be *P*. Then

the interest for the first year $= P \times \frac{4}{100} = P \times (.04)$

The total amount at the end of the first year $P + P \times (.04) = P(1+.04)$

The interest for the second year $= [P(1+0.4) \times (.04)]$ and the total amount at the end of second year $= [P(1+0.4)] + [P(1+0.4)] \times (0.4)$

$= P(1+.04)(1+.04) = P(1+.04)^2$

Similarly the total amount at the end of third year $= P(1+.04)^3$

Thus the sequence for total amounts at the end of successive years is

$P(1+.04), P(1+.04)^2, P(1+.04)^3, \ldots$

The amount at the end of the fifth year is the fifth term of the above gemoetric sequence, that is

$a_5 = [P(1+.04)](1+.04)^{5-1} \quad (\because\ a_5 = a_1 r^{5-1} \text{ and } a_1 = P(1+.04))$

$= P(1+.04)^5$

As *P* = 2000, so the required total amount in rupees = 2000 × (1+.04)^5

$\simeq 2000 \times (1.216653) \simeq 2433.31$

Example 3: The population of a big town is 972405 at present and four years before it was 800,000. Find its rate of increase if it increased geometrically.

Solution: Let the rate of increase be *r* % annually. Then the sequence of population is

$$800,000, 800,000\left(1+\frac{r}{100}\right), 800,000 \times \left(1+\frac{r}{100}\right)^2, \ldots$$

and its fifth term = 972405.

In this case we have,

$$a_n = a_1\left(1 \quad \frac{r}{100}\right)^{n-1} \qquad \left(\because \text{ ratio is } \left(1 \quad \frac{r}{100}\right)\right)$$

Thus $972405 = 800{,}000\left(1 \quad \frac{r}{100}\right)^{5-1} = (\because a_5 \quad 972405 \text{ and } a_1 \quad 800{,}000)$

or $\left(1+\frac{r}{100}\right)^4 = \frac{972405}{800{,}000}$

i.e. $\left(1+\frac{r}{100}\right)^4 = \frac{194481}{160000} \Rightarrow \left(1+\frac{r}{100}\right)^4 = \left(\frac{21}{20}\right)^4 \Rightarrow 1+\frac{r}{100} = \frac{21}{20}$

$$\Rightarrow \frac{r}{100} = \frac{21}{20} - 1 = \frac{1}{20}$$

$$\Rightarrow r = 5$$

Hence the rate of increase is 5%.

Exercise 6.9

1. A man deposits in a bank Rs. 8 in the first year, Rs. 24 in the second year Rs.72 in the third year and so on. Find the amount he will have deposited in the bank by the fifth year.
2. A man borrows Rs. 32760 without interest and agrees to repay the loan in installments, each installment being twice the preceding one. Find the amount of the last installment, if the amount of the first installment is Rs.8.
3. The population of a certain village is 62500. What will be its population after 3 years if it increases geometrically at the rate of 4% annually?
4. The enrollment of a famous school doubled after every eight years from 1970 to 1994. If the enrollment was 6000 in 1994, what was its enrollment in 1970?
5. A singular cholera bacteria produces two complete bacteria in $\frac{1}{2}$ hour. If we start with a colony of a bacteria, how many bacteria will we have in n hours?
6. Joining the mid points of the sides of an equilateral triangle, an equilateral triangle having half the perimeter of the original triangle is obtained. We form a sequence of nested equilateral triangles in the manner described above with the original triangle having

perimeter $\frac{3}{2}$. What will be the total perimeter of all the triangles formed in this way?

6.12 Harmonic Progression (H.P)

A sequence of numbers is called a Harmonic Sequence or Harmonic Progression if the reciprocals of its terms are in arithmetic progression. The sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ is a harmonic sequence since their
reciprocals 1,3,5,7 are in A.P.

Remember that the reciprocal of zero is not defined, so zero can not be the term of a harmonic sequence.

The general form of a harmonic sequence is taken as:

$$\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \ldots \qquad \text{whose } n\text{th term is } \frac{1}{a_1+(n-1)d}$$

Example 1: Find the nth and 8th terms of H.P ; $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$

Solution: The reciprocals of the terms of the sequence,

$\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ are 2,5,8,...

The numbers 2,5,8,... are A.P., so

$a_1 = 2$ and $d = 5 - 2 = 3$.

Putting these values in $a_n = a_1 + (n-1)d$, we have

$$a_n = 2 + (n-1)3$$
$$= 3n - 1$$

Thus the nth term of the given sequence $= \frac{1}{a_n} = \frac{1}{3n-1}$

and substituting $n = 8$ in $\frac{1}{3n-1}$, we get the 8th term of the given H.P. which is $\frac{1}{3\times 8-1} = \frac{1}{23}$

.

Alternatively, a_8 of the A.P. $= a_1 + (8-1)d$

$$= 2 \quad (7).3 \quad 23$$

Thus the 8th term of the given H.P. $= \frac{1}{23}$

Example 2: If the 4th term and 7th term of an H.P. are $\frac{2}{13}$ and $\frac{2}{25}$ respectively, find the sequence.

Solution: Since the 4th term of the H.P. $= \frac{2}{13}$ and its 7th term $= \frac{2}{25}$, therefore the 4th and 7th terms of the corresponding A.P. are $\frac{13}{2}$ and $\frac{25}{2}$ respectively.

Now taking a_1, the first term and *d,* the common difference of the corresponding A.P, we have,

$$a_1 + 3d = \frac{13}{2} \qquad \text{(i)}$$

and $$a_1 + 6d = \frac{25}{2} \qquad \text{(ii)}$$

Subtracting (i) from (ii), gives

$$3d = \frac{25}{2} - \frac{13}{2} = 6 \Rightarrow d = 2$$

From (i), we get

$$a_1 = \frac{13}{2} - 3d = \frac{13}{2} - 6 = \frac{1}{2}$$

Thus a_2 of the A.P. $a_1 + d = \frac{1}{2} + 2 = \frac{5}{2}$

and a_3 of the A.P. $a_1 + 2d = \frac{1}{2} + 2(2)$

$$= \frac{1}{2} + 4 = \frac{9}{2}.$$

Hence the required H.P. is $\frac{2}{1}, \frac{2}{5}, \frac{2}{9}, \frac{2}{13}, \ldots$

6.12.1 Harmonic Mean : A number *H* is said to be the harmonic mean (H.M) between two numbers *a* and *b* if *a*, *H*, *b* are in H.P.

Let *a*, *b* be the two numbers and *H* be their H.M. Then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

therefore, $\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{b}}{2} = \frac{\frac{b+a}{ab}}{2} = \frac{a+b}{2ab}$

and $H = \frac{2ab}{a+b}$

For example, H.M. between 3 and 7 is

$$\frac{2\times 3\times 7}{3+7} = \frac{2\times 21}{10} = \frac{21}{5}$$

6.12.2 *n* Harmonic Means between two numbers

$H_1, H_2, H_3, \ldots, H_n$ are called *n* harmonic means (H.Ms) between *a* and *b* if $a, H_1, H_2, H_3, \ldots H_n, b$ are in H.P. If we want to insert *n* H.Ms. between *a* and *b*, we first find *n* A.Ms. $A_1, A_2, \ldots, A_n$ between $\frac{1}{a}$ and $\frac{1}{b}$, then take their reciprocals to get *n* H.Ms between *a* and *b*, that is, $\frac{1}{A_1}, \frac{1}{A_2}, \ldots, \frac{1}{A_n}$ will, be the required *n* H.Ms. between *a* and *b*.

Example 3: Find three harmonic means between $\frac{1}{5}$ and $\frac{1}{17}$.

Solution: Let A_1, A_2, A_3 be three A.Ms. between 5 and 17, that is, $5, A_1, A_2, A_3, 17$ are in A.P.

Using $a_n = a_1 + (n-1)d$, we get,

$17 = 5 + (5-1)d \qquad (\because\ a_5 = 17 \text{ and } a_1 = 5)$

$4d = 12$

$\Rightarrow \quad d = 3$

Thus $A_1 = 5+3 = 8, A_2 = 5+2(3) = 11$ and $A_3 = 5+3(3) = 14$

Hence $\frac{1}{8}, \frac{1}{11}, \frac{1}{14}$ are the required harmonic means.

Example 4: Find *n* H.Ms between *a* and *b*

Solution: Let $A_1, A_2, A_3, \ldots, A_n$, be *n* A.Ms between $\frac{1}{a}$ and $\frac{1}{b}$.

Then $\frac{1}{a}, A_1, A_2, A_3, \ldots, A_n, \frac{1}{b}$ are in A.P.

Using $a_n = a_1 + (n-1)d$, we get,

$$\frac{1}{b} = \frac{1}{a} + (n+2-1)d$$

or $(n+1)d = \frac{1}{b} - \frac{1}{a} \Rightarrow d = \frac{a-b}{ab(n+1)}$

Thus $A_1 = \frac{1}{a} + d = \frac{1}{a} + \frac{a-b}{ab(n+1)} = \frac{b(n+1)+(a-b)}{ab(n+1)} = \frac{nb+a}{ab(n+1)}$

$$A_2 = \frac{1}{a} + 2d = \frac{1}{a} + 2\left(\frac{a-b}{ab(n+1)}\right) = \frac{b(n+1)+2(a-b)}{ab(n+1)} = \frac{(n-1)b+2a}{ab(n+1)}$$

$$A_3 = \frac{1}{a} + 3d = \frac{1}{a} + 3\left(\frac{a-b}{ab(n+1)}\right) = \frac{b(n+1)+3(a-b)}{ab(n+1)} = \frac{(n-2)b+3a}{ab(n+1)}$$

$$\vdots$$

$$A_n = \frac{1}{a} + nd = \frac{1}{a} + n\left(\frac{a-b}{ab(n+1)}\right) = \frac{b(n+1)+n(a-b)}{ab(n+1)} = \frac{b+na}{ab(n+1)}$$

Hence *n* H.Ms between *a* and *b* are:

$$\frac{ab(n+1)}{nb+a}, \frac{ab(n+1)}{(n-1)b+2a}, \frac{ab(n+1)}{(n-2)b+3a}, \ldots, \frac{ab(n+1)}{b+na}$$

6.13 Relations between Arithmetic, Geometric and Harmonic Means

We know that for any two numbers *a* and *b*

$$A = \frac{a+b}{2}, G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

We first find $A \times H$ that is,

$$A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab$$

$$= G^2$$

Thus *A*, *G*, *H* are in G.P. For example, if

a = –1 and *b* = 5, then

$$A = \frac{-1+5}{2} = 2, \qquad G = \pm\sqrt{-1\times 5} = \pm\sqrt{5}i$$

$$H = \frac{2(-1).5}{-1+5} = \frac{-10}{4} = \frac{-5}{2}$$

$$A \times H = 2 \times \frac{-5}{2} = -5 \text{ and } G^2 = (\pm\sqrt{5}i)^2 = 5i^2 = -5$$

It follows that $A \times H = G^2$ and *A*, *G*, *H* are in G.P.

Note: $G^2 = A \times H$ even if $a, b \in C$

Now we show that $A > H$ for any two distinct positive real numbers.

$$A > H \text{ if } \frac{a+b}{2} > \frac{2ab}{a+b}$$

or $\quad (a+b)^2 > 4ab$

or $\quad (a+b)^2 - 4ab > 0 \Rightarrow (a-b)^2 > 0$

which is true because $a - b$ is a real number and the square of a real number is always positive.

Also $A > G$ if a, b are any two distinct positive real numbers.

$$A > G \text{ if } \frac{a+b}{2} > \pm\sqrt{ab}$$

or $\quad a + b \mp 2\sqrt{ab} > 0$

$\Rightarrow \quad (\sqrt{a} \mp \sqrt{b})^2 > 0$

which is true because $\sqrt{a} \mp \sqrt{b}$ are non zero real numbers and the squares of real numbers are always positive.

Now we prove that

i) $A > G > H$ if a, b are any two distinct positive real numbers and $G = \sqrt{ab}$.

ii) $A < G < H$ if a, b are any two distinct negative real numbers and $G = -\sqrt{ab}$.

To prove (i) we first show that $A > G$, i.e.,

$$A > G \text{ if } \frac{a+b}{2} > \sqrt{ab}$$

$\Rightarrow \quad (\sqrt{a} - \sqrt{b})^2 > 0$

which is true (write the missing steps as given above)

Thus $A > G$ (1)

Again $G > H$,

if $\sqrt{ab} > \dfrac{2ab}{a+b}$

or $a + b > 2\sqrt{ab}$

$\Rightarrow \quad a + b - 2\sqrt{a}\sqrt{b} > 0$

$\Rightarrow \quad (\sqrt{a} - \sqrt{b})^2 > 0$

which is true since $\sqrt{a} - \sqrt{b}$ is a real number.

Thus $G > H$ (2)

From (1) and (2), we have

$A > G > H$

To prove (ii), we show that

$A < G$ if

$$\frac{a+b}{2} < -\sqrt{ab}$$

Let $a = -m$ and $b = -n$ where m and n are positive real numbers. Then

$$\frac{-m-n}{2} < -\sqrt{(-m)(-n)}$$

or $-\dfrac{m+n}{2} < -\sqrt{mn} \Rightarrow \dfrac{m+n}{2} > \sqrt{mn}$

$\Rightarrow (\sqrt{m} - \sqrt{n})^2 > 0$ (See part(i))

which is true, that is,

$A < G$

Similarly, we can prove that

$G < H$

Hence $A < G < H$

Exercise 6.10

1. Find the 9th term of the harmonic sequence

i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ ii) $\frac{-1}{5}, \frac{-1}{3}, -1, \ldots$

2. Find the 12th term of the following harmonic sequences

i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ ii) $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \ldots$

3. Insert five harmonic means between the following given numbers,

i) $\frac{-2}{5}$ and $\frac{2}{13}$ ii) $\frac{1}{4}$ and $\frac{1}{24}$

4. Insert four harmonic means between the following given numbers.

i) $\frac{1}{3}$ and $\frac{1}{23}$ ii) $\frac{7}{3}$ and $\frac{7}{11}$ iii) 4 and 20

5. If the 7th and 10th terms of an H.P. are $\frac{1}{3}$ and $\frac{5}{21}$ respectively, find its 14th term.

6. The first term of an H.P. is $-\frac{1}{3}$ and the fifth term is $\frac{1}{5}$. Find its 9th term.

7. If 5 is the harmonic mean between 2 and *b*, find *b*.

8. If the numbers $\frac{1}{k}, \frac{1}{2k+1}$ and $\frac{1}{4k-1}$ are in harmonic sequence, find k.

9. Find *n* so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be H.M. between *a* and *b*.

10. If a^2, b^2 and c^2 are in A.P. show that $a + b$, $c + a$ and $b + c$ are in H.P.

11. The sum of the first and fifth term of the harmonic sequence is $\frac{4}{7}$, if the first term is $\frac{1}{2}$, find the sequence.

12. If *A*, *G* and *H* are the arithmetic, geometric and harmonic means between *a* and *b* respectively, show that $G^2 = AH$.

13. Find *A, G, H* and show that $G^2 = AH$. if

i) $a = -2, b \quad 6$ ii) $a = 2i, b = 4i$ iii) $a = 9, b = 4$

14. Find *A, G, H* and verify that $A > G > H$ $(G > 0)$, if

i) $a = 2, b = 8$ ii) $a = \frac{2}{5}, b = \frac{8}{5}$

15. Find *A, G, H* and verify that $A < G < H$ $(G < 0)$, if

i) $a = -2, b \quad 8$ ii) $a = \frac{-2}{5}, b = \frac{-8}{5}$

16. If the H.M and A.M. between two numbers are 4 and $\frac{9}{2}$ respectively, find the numbers.

17. If the (positive) G.M. and H.M. between two numbers are 4 and $\frac{16}{5}$, find the numbers.

18. If the numbers $\frac{1}{2}, \frac{4}{21}$ and $\frac{1}{36}$ are subtracted from the three consecutive terms of a G.P., the resulting numbers are in H.P. Find the numbers if their product is $\frac{1}{27}$.

6.14 Sigma Notation (or Summation Notation)

The Greek letter $\sum$(sigma) is used to denote sums of different types . For example the notation $\sum_{i=m}^{n} a_i$ is used to express the sum

$$a_m + a_{m+1} + a_{m+2} + ... + a_n$$ and the sum expression

1 + 3 + 5+ ....to *n* terms.

is written as $\sum_{k=1}^{n}(2k-1)$,

where $(2k-1)$ is the *k*th term of the sum and *k* is called the index of summation. '1' and *n* are called the lower limit and the upper limit of summation respectively. The sum of the first *n* natural numbers, the sum of squares of the first *n* natural numbers and the sum of the cubes of the first *n* natural numbers are expressed in sigma notation as:

$$1+2+3+....+n = \sum_{k=1}^{n} k$$

$$1^2+2^2+3^2+....+n^2 = \sum_{k=1}^{n} k^2$$

$$1^3+2^3+3^3+....+n^3 = \sum_{k=1}^{n} k^3$$

We evaluate $\sum_{k=1}^{n}[k^m-(k-1)^m]$ for any positive integer *m* and shall use this result to find out

formulas for three expressions stated above.

$$\sum_{k=1}^{n}[k^m-(k-1)^m]=(1^m-0^m)+(2^m-1^m)+(3^m-2^m)+\ldots.$$
$$+[(n-1)^m-(n-2)^m]+[n^m-(n-1)^m]=n^m$$

i.e., $\sum_{k=1}^{n}[(k^m-(k-1)^m]=n^m$

If $m=1$,

then $\sum_{k=1}^{n}[(k^1-(k-1)^1]=n^1$

i.e. $\sum_{k=1}^{n}1=n$

6.15 To Find the Formulae for the Sums

i) $\sum_{k=1}^{n}k$ ii) $\sum_{k=1}^{n}k^2$ iii) $\sum_{k=1}^{n}k^3$

i) We know that $k^2-(k-1)^2 = 2k-1$ (A)

Taking summation on both sides of (A) from k = 1 to n, we have

$$\sum_{k=1}^{n}[k^2-(k-1)^2]=\sum_{k=1}^{n}(2k-1)$$

i.e., $n^2=2\sum_{k=1}^{n}k - n \qquad (\because \sum_{k=1}^{n}1 = n)$

or $2\sum_{k=1}^{n}k=n^2+n$

Thus $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$

ii) Consider the identity

$k^3-(k-1)^3 = 3k^2-3k+1$ (B)

Taking summation of (B) on both sides from $k=1$ to n, we get

$$\sum_{k=1}^{n}[k^3-(k-1)^3]=\sum_{k=1}^{n}(3k^2-3k+1)$$

i.e., $n^3=3\sum_{k=1}^{n}k^2-3\sum_{k=1}^{n}k+n$

or $3\sum_{k=1}^{n}k^2=n^3-n+3\sum_{k=1}^{n}k$

$=n(n+1)(n-1)+3\times\frac{n(n+1)}{2}$

$=n(n+1)\left[n-1+\frac{3}{2}\right]=\frac{n(n+1)(2n+1)}{2}$

Thus $\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$

iii) We know that $(k-1)^4 = k^4-4k^3+6k^2-4k+1$ and this identity can be written as: (C)

$k^4-(k-1)^4=4k^3-6k^2+4k-1$

Taking summation on both sides of (C), from $k=1$ to n, we get,

$$\sum_{k=1}^{n}[k^4-(k-1)^4]=\sum_{k=1}^{n}(4k^3-6k^2+4k-1)$$

i.e., $n^4=4\sum_{k=1}^{n}k^3-6\sum_{k=1}^{n}k^2+4\sum_{k=1}^{n}k-n$

or $4\sum_{k=1}^{n}k^3=n^4+n+6\sum_{k=1}^{n}k^2-4\sum_{k=1}^{n}k$

$=n(n+1)(n^2-n+1)+6\times\frac{n(n+1)(2n+1)}{6}-4\times\frac{n(n+1)}{2}$

$=n(n+1)[n^2-n+1+2n+1-2]$

$=n(n+1)(n^2+n)=n(n+1).n(n+1)$

Thus $\sum_{k=1}^{n}k^3=\frac{[n(n+1)]^2}{4}=\left[\frac{n(n+1)}{2}\right]^2$

Example 1: Find the sum of the series $1^3+3^3+5^3+\ldots$ to n terms

Solution: $T_k=(2k-1)^3 \qquad (\because 1+(k-2)2=2k-1)$

$$=8k^3-12k^2+6k-1$$

Let S_n denote the sum of n terms of the given series, then

$$S_n=\sum_{k=1}^{n} T_k$$

or $\quad S_n=\sum_{k=1}^{n}(8k^2-12k^2+6k-1)$

$$=8\sum_{k=1}^{n}k^3-12\sum_{k=1}^{n}k^2+6\sum_{k=1}^{n}k-\sum_{k=1}^{n}1$$

$$=8\left[\frac{n(n+1)}{2}\right]^2-12\left[\frac{n(n+1)(2n+1)}{6}\right]+6\left[\frac{n(n+1)}{2}\right]-n$$

$$=2n^2(n+1)^2-2n(n+1)(2n+1)+3n(n+1)-n$$

$$=2n^2(n^2+2n+1)-2n(2n^2+3n+1)+n(3n+3)-n$$

$$=2n[(n^3+2n^2+n)-(2n^2+3n+1)]+n(3n+3-1)$$

$$=2n[n^3-2n-1]+n(3n+2)$$

$$=n[2n^3-4n-2+3n+2]$$

$$=n[2n^2-n]=n.[n(2n^2-1)]$$

$$=n^2[2n^2-1]$$

Example 2: Find the sum of n terms of series whose nth terms is $n^3+\frac{3}{2}n^2+\frac{1}{2}n+1$

Solution: Given that

$$T_n=n^3+\frac{3}{2}n^2+\frac{1}{2}n+1$$

Thus $\quad T_k=k^3+\frac{3}{2}k^2+\frac{1}{2}k+1$

and $\quad S_n=\sum_{k=1}^{n}\left(k^3+\frac{3}{2}k^2+\frac{1}{2}k+1\right)$

$$=\sum_{k=1}^{n}k^3+\frac{3}{2}\sum_{k=1}^{n}k^2+\frac{1}{2}\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

$$=\frac{n^2(n+1)^2}{4}+\frac{3}{2}\times\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\times\frac{n(n+1)}{2}+n$$

$$=\frac{n}{4}[n(n^2+2n+1)+(2n^2+3n+1)+(n+1)+4]$$

$$=\frac{n}{4}(n^3+2n^2+n+2n^2+3n+1+n+1+4)$$

$$=\frac{n}{4}(n^3+4n^2+5n+6)$$

Exercise 6.11

Sum the following series upto n terms.

1. $1\times1+2\times4+3\times7+\ldots$

2. $1\times3+3\times6+5\times9+\ldots$

3. $1\times4+2\times7+3\times10+\ldots$

4. $3\times5+5\times9+7\times13+\ldots$

5. $1^2+3^2+5^2+\ldots$

6. $2^2+5^2+8^2+\ldots$

7. $2\times1^2+4\times2^2+6\times3^2+\ldots$

8. $3\times2^2+5\times3^2+7\times4^2+\ldots$

9. $2\times4\times7+3\times6\times10+4\times8\times13+\ldots$

10.. $1\times4\times6+4\times7\times10+7\times10\times14+\ldots$

11. $1+(1+2)+(1+2+3)+\ldots$

12. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$

13. $2+(2+5)+(2+5+8)+\ldots$

14. Sum the series.

i) $1^2-2^2+3^2-4^2+\ldots+(2n-1)^2-(2n)^2$

ii) $1^2-3^2+5^2-7^2+\ldots+(4n-3)^2-(4n-1)^2$

iii) $\frac{1^2}{1}+\frac{1^2+2^2}{2}+\frac{1^2+2^2+3^2}{3}+\ldots$ to n terms

15. Find the sum to n terms of the series whose nth terms are given.

i) $3n^2+n+1$ $\qquad$ ii) n^2+4n+1

16. Given nth terms of the series, find the sum to $2n$ terms.

i) $3n^2+2n+1$ $\qquad$ ii) n^3+2n+3

Permutation Combination and Probability

7.1 Introduction

The **factorial notation** was introduced by Christian Kramp (1760 - 1826) in 1808. This notation will be frequently used in this chapter as well as in finding the Binomial Coefficients in later chapter. Let us have an introduction of **factorial notation**.

Let n be a positive integer. Then the product $n(n-1)(n-2)\ldots 3 . 2 . 1$ is denoted by $n!$ or $\underline{|n}$ and read as n factorial.

That is, $\boxed{n! = n(n-1)(n-2)\ldots 3.2.1}$

For Example,

1! = 1

	2! = 2.1	= 2	⇒ 2! = 2.1!
	3! = 3.2.1	= 6	⇒ 3! = 3.2!
	4! = 4.3.2.1	= 24	⇒ 4! = 4.3!
	5! = 5.4.3.2.1	= 120	⇒ 5! = 5.4!
and	6! = 6.5.4.3.2.1	= 720	⇒ 6! = 6.5!

Thus for a positive integer n, we define n factorial as:
$n! = n(n-1)!$ where $0! = 1$

Example 1: Evaluate $\frac{8!}{6!}$

Solution: $\frac{8!}{6!} = \frac{8.7.6.5.4.3.2.1}{6.5.4.3.2.1} = 56$

Example 2: Write 8.7.6.5 in the factorial form

Solution: $8.7.6.5 = \frac{8.7.6.5.4.3.2.1}{4.3.2.1} = \frac{8!}{4!}$

Example 3: Evaluate $\frac{9!}{6!3!}$

Solution: $\frac{9!}{6!3!} = \frac{(9.8.7)6!}{6!(3.2.1)} = 84$

or $\frac{9!}{6!3!} = \frac{9.8.7.6.5.4.3.2.1}{6.5.4.3.2.1.3.2.1} = 84$

Exercise 7.1

1. Evaluate each of the following:

i) $4!$ ii) $6!$ iii) $\frac{8!}{7!}$ iv) $\frac{10!}{7!}$

v) $\frac{11!}{4!7!}$ vi) $\frac{6!}{3!3!}$ vii) $\frac{8!}{4!2!}$ viii) $\frac{11!}{2!4!5!}$

ix) $\frac{9!}{2!(9-2)!}$ x) $\frac{15!}{15!(15-15)!}$ xi) $\frac{3!}{0!}$ xii) $4!.0!.1!$

2. Write each of the following in the factorial form:

i) 6.5.4. ii) 12.11.10 iii) 20.19.18.17

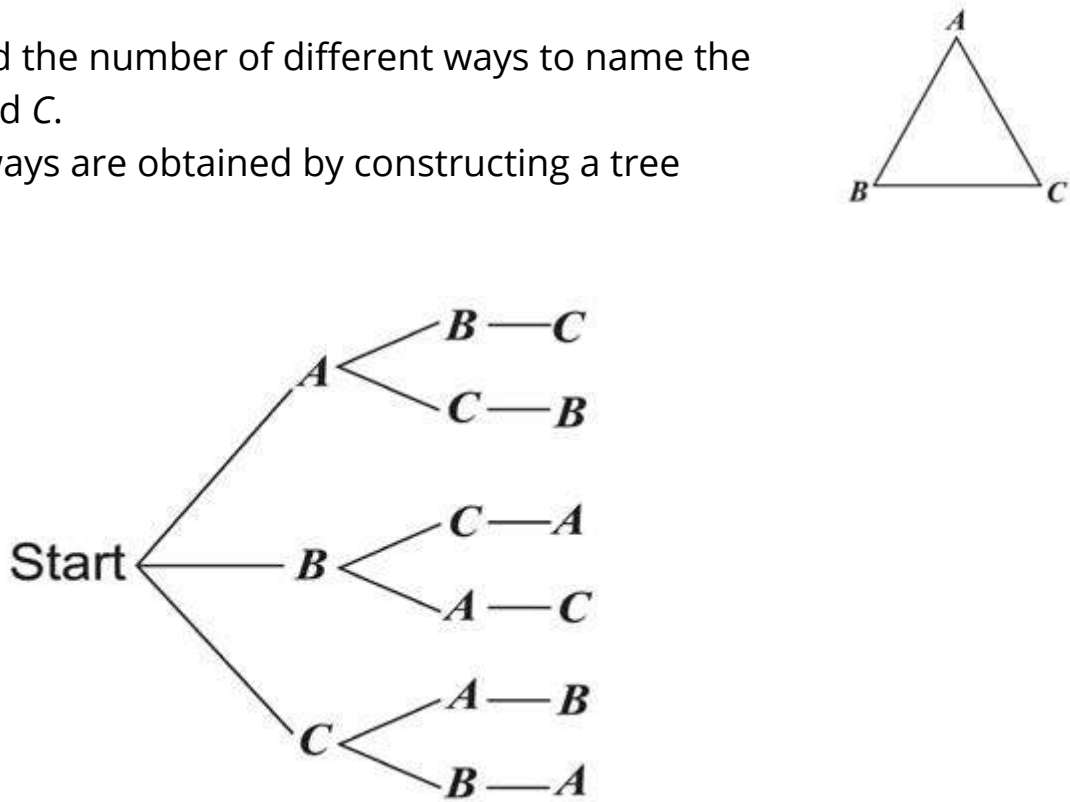
iv) $\frac{10.9}{2.1}$ v) $\frac{8.7.6}{3.2.1}$ vi) $\frac{52.51.50.49}{4.3.2.1}$

vii) $n(n-1)(n-2)$ viii) $(n+2)(n+1)(n)$ ix) $\frac{(n+1)(n)(n-1)}{3.2.1}$

x) $n(n-1)(n-2)\ldots(n-r+1)$

7.2 Permutation

Suppose we like to find the number of different ways to name the triangle with vertices *A*, *B* and *C*.

The various possible ways are obtained by constructing a tree diagram as follows:

To determine the possible ways, we count the paths of the tree, beginning from the start to the end of each branch. So, we get 6 different names of triangle.

ABC, ACB, BCA, BAC, CAB, CBA.

Thus there are six possible ways to write the name of the triangle with vertices *A, B* and *C.*

Explanation: In the figure, we can write any one of the three vertices *A, B, C* at first place. After writing at first place any one of the three vertices, two vertices are left. So, there are two choices to write at second place. After writing the vertices at two places, there is just one vertex left. So, we can write only one vertex at third place.

Another Way of Explanation:

Think of the three places as shown

Since we can write any one of the three vertices at first place, so it is written in 3 different ways as shown. | 3 | | |

Now two vertices are left. So, corresponding to each way of writing at first place, there

are two ways of writing at second place as shown. | 3 | 2 | |

Now just one vertex is left. So, we can write at third place only one vertex in one way as shown. | 3 | 2 | 1 |

The total number of possible ways (arrangements) is the product 3.2.1= 6.

This example illustrates the fundamental principle of counting.

Fundamental Principle of Counting:

Suppose *A* and *B* are two events. The first event *A* can occur in *p* different ways. After *A* has occurred, *B* can occur in *q* different ways. The number of ways that the two events can occur is the product *p.q*.

This principle can be extended to three or more events. For instance, the number of ways that three events *A, B* and *C* can occur is the product *p.q.r*.

One important application of the Fundamental Principle of Counting is to determine the number of ways that *n* objects can be arranged in order. An ordering (arrangement) of *n* objects is called a **permutation** of the objects.

A **permutation** of *n* different objects is an ordering (arrangement) of the objects such that one object is first, one is second, one is third and so on.

According to Fundamental Principle of Counting:

i) Three books can be arranged in a row taken all at a time = 3.2.1 = 3! ways

ii) Number of ways of writing the letters of the **WORD** taken all at a time = 4.3.2.1 = 4!

Each arrangement is called a permutation. Now we have the following definition.

A permutation of *n* different objects taken *r* ($\leq$ *n*) at a time is an arrangement of the *r* objects. Generally it is denoted by ^nP_r or *P*(*n*,*r*).

Prove that: $^np_r = n(n-1)(n-2)....(n-r+1) = \dfrac{n!}{(n-r)!}$

Proof: As there are *n* different objects to fill up *r* places. So, the first place can be filled in *n* ways. Since repetitions are not allowed, the second place can be filled in (*n*–1) ways, the third place is filled in (*n*–2) ways and so on. The *r*th place has $n-(r-1) = n-r+1$ choices to be filled in. Therefore, by the fundamental principle of counting, *r* places can be filled by *n* different objects in $n(n-1)(n-2)\ ....\ (n-r+1)$ ways

$$\therefore \quad p = n(n-1)(n-2)....(n-r+1)$$

$$= \frac{n(n-1)(n-2)....(n-r+1)(n-r)(n-r-1) .... 3.2.1}{(n-r)(n-r-1) .... 3.2.1}$$

$$\Rightarrow {}^nP_r = \frac{n!}{(n-r)!}$$

which completes the proof.

Corollary: If $r = n$, then

$${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

$\Rightarrow$ n different objects can be arranged taken all at a time in $n!$ ways.

Example 1: How many different 4-digit numbers can be formed out of the digits 1, 2, 3, 4, 5, 6, when no digit is repeated?

Solution: The total number of digits = 6

The digits forming each number = 4.

So, the required number of 4-digit numbers is given by:

$${}^6p_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6.5.4.3.2.1}{2.1} = 6.5.4.3 = 360$$

Example 2: How many signals can be made with 4-different flags when any number of them are to be used at a time?

Solution: The number of flags = 4

Number of signals using 1 flag = ${}^4P_1$ = 4

Number of signals using 2 flags = ${}^4P_2$ = 4 . 3 = 1 2

Number of signals using 3 flags = ${}^4P_3$ = 4.3.2 = 24

Number of signals using 4 flags = ${}^4P_4$ = 4.3.2.1 = 24

$\therefore$ Total Number of signals = 4 + 12 + 24 + 24 = 64.

Example 3: In how many ways can a set of 4 different mathematics books and 5 different physics books be placed on a shelf with a space for 9 books, if all books on the same subject are kept together?

Solution: 4 different Mathematics books can be arranged among themselves in 4! ways. 5 different Physics books can be arranged among themselves in 5! ways.To every one way of arranging 4 mathematics books there are 5! ways of arranging 5 physics books. The books in the two subjects can be arranged subject-wise in 2! ways.

So the number of ways of arranging the books as given by.

$$4! \times 5! \times 2! = 4 \times 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1$$
$$= 5740$$

Exercise 7.2

1. Evaluate the following:

i) ${}^{20}P_3$ ii) ${}^{16}P_4$ iii) ${}^{12}P_5$ iv) ${}^{10}P_7$ v) ${}^{9}P_8$

2. Find the value of n when:

i) ${}^nP_2 = 30$ ii) ${}^{11}P_n = 11.10.9$ iii) ${}^nP_4 : {}^{n-1}P_3 = 9:1$

3. Prove from the first principle that:

i) ${}^nP_r = n.{}^{n-1}P_{r-1}$ ii) ${}^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$

4. How many signals can be given by 5 flags of different colours, using 3 flags at a time?

5. How many signals can be given by 6 flags of different colours when any number of flags can be used at a time?

6. How many words can be formed from the letters of the following words using all letters when no letter is to be repeated:

i) PLANE ii) OBJECT iii) FASTING?

7. How many 3-digit numbers can be formed by using each one of the digits 2, 3, 5, 7, 9 only once?

8. Find the numbers greater than 23000 that can be formed from the digits 1, 2, 3, 5, 6 , without repeating any digit.

HINT: The first two digits on L.H.S. will be 23 etc.

9. Find the number of 5-digit numbers that can be formed from the digits 1, 2, 4, 6, 8 (when no digit is repeated), but

i) the digits 2 and 8 are next to each other;

ii) the digits 2 and 8 are not next to each other.

10. How many 6-digit numbers can be formed, without repeating any digit from the digits 0, 1, 2, 3, 4, 5? In how many of them will 0 be at the tens place?

11. How many 5-digit multiples of 5 can be formed from the digits 2, 3, 5, 7,9, when no digit is repeated.

12. In how many ways can 8 books including 2 on English be arranged on a shelf in such a way that the English books are never together?

13. Find the number of arrangements of 3 books on English and 5 books on Urdu for placing them on a shelf such that the books on the same subject are together.

14. In how many ways can 5 boys and 4 girls be seated on a bench so that the girls and the boys occupy alternate seats?

7.2.1 Permutation of Things Not All Different

Suppose we have to find the permutation of the letters of the word BITTER, using all the letters in it. We see that all the letters of the word BITTER are not different and it has 2 Ts in it. Obviously, the interchanging of Ts in any permutation, say BITTER, will not form a new permutation. However, if the two Ts are replaced by T_1 and T_2, we get the following two permutation of BITTER

$$BIT_1T_2ER \text{ and } BIT_2T_1ER$$

Similarly, the replacement of the two Ts by T_1 and T_2 in any other permutation will give rise to 2 permutation.

Now, BIT_1T_2ER consists of 6 different letters which can be permuted among themselves in 6 ! different ways. Hence the number of permutation of the letters of the word BITTER taken all at a time

$$=\frac{6!}{2}=\frac{6.5.4.3.2.1}{2} \quad 360$$

This example guides us to discover the method of finding the permutation of *n* things all of which are not different. Suppose that out of *n* things, n_1 are alike of one kind and n_2 are

alike of second kind and the rest of them are all different. Let x be the required number of permutation. Replacing n_1 alike things by n_1 different things and n_2 alike things by n_2 different things, we shall get all the *n* things distinct from each other which can be permuted among themselves in *n*! ways. As n_1 different things can be permuted among themselves in $(n_1)!$ ways and n_2 different things can be arranged among themselves in $(n_2)!$ ways, so because of the replacement suggested above, x permutation would increase to $x \times (n_1)! \times (n_2)!$ number of ways.

$$\therefore \quad x \times (n_1)! \times (n_2)! = (n)!$$

Hence $$x=\frac{(n)!}{(n_1)! \times (n_2)!}=\binom{n}{n_1, n_2}$$

Cor. If there are n_1 alike things of one kind, n_2 alike things of second kind and n_3 alike things of third kind, then the number of permutation of *n* things,taken all at a time is given by:

$$\frac{n!}{(n_1)! \times (n_2)! \times (n_3)!} = \binom{n}{n_1, n_2 n_3}$$

Example 1: In how many ways can be letters of the word MISSISSIPPI be arranged when all the letters are to be used?

Solution: Number of letters in MISSISSIPPI $= 11$

In MISSISSIPPI,

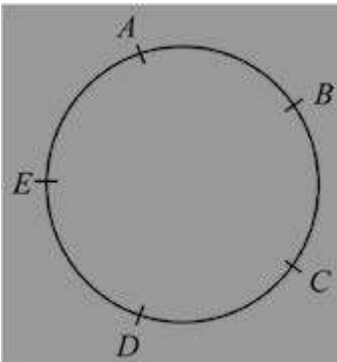
I is repeated 4 times

S is repeated 4 times

P is repeated 2 times

M comes only once.

Required number of permutation $=\binom{11}{4,4,2,1}$

$$= \frac{(11)!}{(4)! \times (4)! \times (2)! \times (1)!} \quad 34650 \text{ ways}$$

7.2.2 Circular Permutation

So far we have been studying permutation of things which can be represented by the points on a straight line. We shall now study the permutation of things which can be represented by the points on a circle. The permutation of things which can be represented by the points on a circle are called **Circular Permutation**.

The method of finding circular permutation is illustrated by the following examples.

Example 2: In how many ways can 5 persons be seated at a round table.

Solution: Let A, B, C, D, E be the 5 persons One of the ways of seating them round a table is shown in the adjoining figure. If each person moves one or two or more places to the left or the right, they will, no doubt, be occupying different seats, but their positions relative to each other will remain the same.

So, when A occupies a certain seat, the remaining 4 persons will be permuting their seats among themselves in 4! ways.

Hence the number of arrangements = 4! = 24

Example 3: In how many ways can a necklace of 8 beads of different colours be made?

Solution: The number of beads = 8

The number of arrangements of 8 beads in the necklace will be like the seating of 8 persons round a table.

$\Rightarrow$ The number of such necklaces (fixing one of the beads) = 7!

Now suppose the beads are a, b, c, d, e , f g , h and the necklace is as shown in Fig. (i) below:

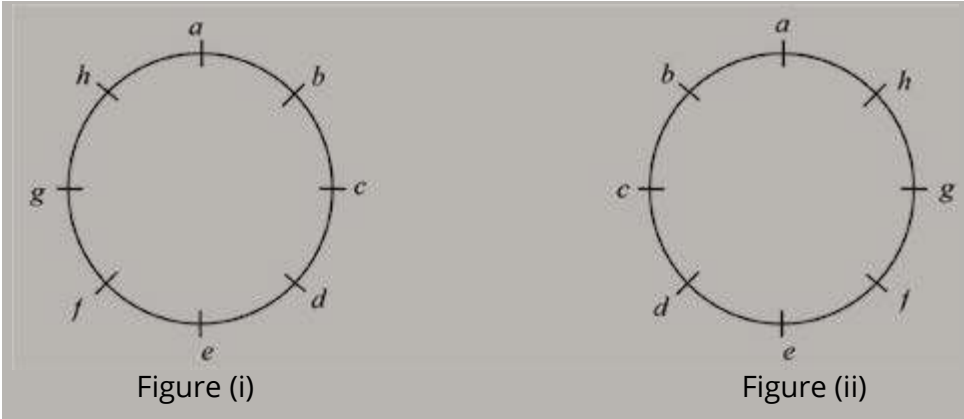



Figure (i) Figure (ii)

By flipping the necklace we get the necklace as shown in figure (ii). We observe that the two arrangements of the beads are actually the same.

Hence the required number of necklaces = $= \frac{1}{2} \times (7)! = 2520$

Exercise 7.3

1. How many arrangements of the letters of the following words, taken all together, can be made:
   i) PAKPATTAN ii) PAKISTAN
   ii) MATHEMATICS iv) ASSASSINATION?
2. How many permutation of the letters of the word PANAMA can be made, if P is to be the first letter in each arrangement?
3. How many arrangements of the letters of the word ATTACKED can be made, if each arrangement begins with C and ends with K?
4. How many numbers greater than 1000,000 can be formed from the digits 0, 2, 2,2, 3, 4 ,4 ?
5. How many 6-digit numbers can be formed from the digits 2, 2, 3, 3, 4, 4? How many of them will lie between 400,000 and 430,000?
6. 11 members of a club form 4 committees of 3, 4, 2, 2 members so that no member is a member of more than one committee. Find the number of committees.
7. The D.C.Os of 11 districts meet to discuss the law and order situation in their districts. In how many ways can they be seated at a round table, when two particular D.C.Os insist on sitting together?
8. The Governor of the Punjab calls a meeting of 12 officers. In how many ways can they be seated at a round table?

9. Fatima invites 14 people to a dinner. There are 9 males and 5 females who are seated at two different tables so that guests of one sex sit at one round table and the guests of the other sex at the second table. Find the number of ways in which all gests are seated.

10. Find the number of ways in which 5 men and 5 women can be seated at a round table in such a way that no two persons of the same sex sit together.

11. In how many ways can 4 keys be arranged on a circular key ring?

12. How many necklaces can be made from 6 beads of different colours?

7.3 Combination

While counting the number of possible permutation of a set of objects, the order is important. But there are situations where order is immaterial. For example

i) $ABC, ACB, BAC, BCA, CAB, CBA$ are the six names of the triangle whose vertices are *A, B* and *C*. We notice that inspite of the different arrangements of the vertices of the triangle, they represent one and the same triangle.

ii) The 11 players of a cricket team can be arranged in 11! ways, but they are players of the same single team. So, we are interested in the membership of the committee (group) and not in the way the members are listed (arranged). Therefore, a combination of n different objects taken *r* at a time is a set of *r* objects.

The number of combinations of *n* different objects taken *r* at a time is denoted by ^nC_r or $C(n, r)$ or $\binom{n}{r}$ and is given by

$$^nc_r = \frac{n!}{r!(n-r)!}$$

Proof: There are ^nC_r combinations of *n* different objects taken *r* at a time. Each combination consists of *r* different objects which can be permuted among themselves in *r*! ways. So, each combination will give rise to *r*! permutation. Thus there will be ^nC_r x *r*! permutation of *n* different objects taken *r* at a time.

$$^nc_r \times r! = {}^np_r$$

$$\Rightarrow \quad {}^nc_r \times r = \frac{n!}{(n-r)!} \therefore {}^nc_r = \frac{n!}{r!(n-r)!}$$

Which completes the proof.

Corollary:

i) If $r = n$, then $\quad {}^nc_n = \frac{n!}{n!(n-r)!} = \frac{n!}{n!0!} = 1$

ii) If $r = 0$, then $\quad {}^nc_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{0!n!} = 1$

7.3.1 Complementary Combination

Prove that: $^nC_r = {}^nC_{n-r}$

Proof: If from *n* different objects, we select *r* objects then $(n - r)$ objects are left.

Corresponding to every combination of *r* objects, there is a combination of $(n - r)$ objects and vice versa.

Thus the number of combinations of *n* objects taken *r* at a time is equal to the number of combinations of *n* objects taken $(n - r)$ at a time.

$$\therefore \quad {}^nC_r = {}^nC_{n-r}$$

Other wise: $^nc_{n-r} = \dfrac{n!}{(n-r)!(n-n+r)!}$

$$= \frac{n!}{(n-r)!r!} = \frac{n!}{r!(n-r)!}$$

$$\Rightarrow {}^nc_{n-r} = {}^nc_r$$

Note: This result will be found useful in evaluating ^nC_r when $r > \frac{n}{2}$.

e.g $^{12}C_{10} = {}^{12}C_{12-10} = {}^{12}c_2 = \frac{(12).(11)}{2} = (6).(11) = 66$

Example 1: If ${}^nC_8 = {}^nC_{12}$, find n.

Solution: We know that ${}^nC_r = {}^nC_{n-r}$

$\therefore \quad {}^nC_8 = {}^nC_{n-8} \qquad$ (i)

But it is given that ${}^nC_8 = {}^nC_{12} \qquad$ (ii)

From (i) and (ii), we conclude that

${}^nC_{n-8} = {}^nC_{12}$

$\Rightarrow \quad n - 8 = 12$

$\therefore \quad n = 20$

Example 2: Find the number of the diagonals of a 6-sided figure.

Solution: A 6-sided figure has 6 vertices. Joining any two vertices we get a line segment.

$\therefore$ Number of line segments ${}^6C_2 = \frac{6!}{2!4!} = 15$

But these line segments include 6 sides of the figure

$\therefore$ Number of diagonals = 1 5 - 6 = 9

Example 3: Prove that: ${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r$

Solution: L.H.S. $= {}^{n-1}C_r + {}^{n-1}C_{r-1}$

$$= \frac{\underline{|n-1}}{\underline{|r}\,\underline{|n-1-r}} + \frac{\underline{|n-1}}{\underline{|r-1}\,\underline{|n-r}}$$

$$= \frac{\underline{|n-1}}{r\underline{|r-1}\,\underline{|n-r-1}} + \frac{\underline{|n-1}}{\underline{|r-1}(n-r)\underline{|n-r-1}}$$

$$= \frac{\underline{|n-1}}{\underline{|r-1}\,\underline{|n-r-1}}\left[\frac{1}{r} + \frac{1}{n-r}\right] = \frac{\underline{|n-1}}{\underline{|r-1}\,\underline{|n-r-1}}\left[\frac{n-r+r}{r(n-r)}\right]$$

$$= \frac{n\underline{|n-1}}{r\underline{|r-1}(n-r)\underline{|n-r-1}} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \quad {}^nC_r$$

= R.H.S.

Hence ${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r$

Exercise 7.4

1. Evaluate the following:

i) ${}^{12}C_3$ ii) ${}^{20}C_{17}$ iii) ${}^nC_4$

2. Find the value of n, when

i) ${}^nC_5 = {}^nC_4$ ii) ${}^nC_{10} = \frac{12 \times 11}{2!}$ iii) ${}^nC_{12} = {}^nC_6$

3. Find the values of n and r, when

i) ${}^nC_r = 35$ and ${}^nP_r = 210$

ii) ${}^{n-1}C_{r-1} : {}^nC_r : {}^{n+1}C_{r+1} = 3 : 6 : 1\ 1$

4. How many (*a*) diagonals and (*b*) triangles can be formed by joining the vertices of the polygon having:

i) 5 sides ii) 8 sides iii) 12 sides?

5. The members of a club are 12 boys and 8 girls. In how many ways can a committee of 3 boys and 2 girls be formed?

6. How many committees of 5 members can be chosen from a group of 8 persons when each committee must include 2 particular persons?

7. In how many ways can a hockey team of 11 players be selected out of 15 players? How many of them will include a particular player?

8. Show that: ${}^{16}C_{11} + {}^{16}C_{10} = {}^7C_{11}$

9. There are 8 men and 10 women members of a club. How many committees of can be formed, having;

i) 4 women ii) at the most 4 women iii) atleast 4 women?

10. Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

7.4 Probability

We live in an uncertain world where very many events cannot be predicted with complete certainty, e.g.

i) In a cloudy weather, we cannot be sure whether it will or will not rain. However,

we can say that there is 1 chance out of 2 that the rain will fall.

ii) There are 6 theorems on circle out of which one theorem is asked in the Secondary School Examination. Evidently, there is **1 chance out of 6** that a particular theorem will be asked in the examination.

In simple situations, we are guided by our experience or intuition. However, we cannot be sure about our predictions. Nevertheless, in more complex situations, we cannot depend upon guess work and we need more powerful tools for analyzing the situations and adopting the safer path for the achievement of our goals.

In order to guide in solving complex problems of everyday life, two French Mathematicians, BLAISE PASCAL (1623-62) and PIERRE DE FERMAT (1601 - 65), introduced **probability theory**. A very simple definition of probability is given below:

Probability is the numerical evaluation of a chance that a particular event would occur.

This definition is too vague to be of any practical use in estimating the chance of the occurrence of a particular event in a given situation. But before giving a comprehensive definition of probability we must understand some terms connected with probability.

Sample Space and Events: The set S consisting of all possible outcomes of a given experiment is called the **sample space**. A particular outcome is called an **event** and usually denoted by *E*. An event *E* is a subset of the sample space *S*. For example,

i) In tossing a fair coin, the possible outcomes are a Head (*H*) or a Tail (*T*) and it is written as: $S = \{H, T\} \Rightarrow n(S) = 2$.

ii) In rolling a die the possible outcomes are 1 dot, 2 dots, 3 dots, 4 dots, 5 dots or 6 dots on the top.

$\therefore \quad S = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6$

To get an even number 2, 4 or 6 is such event and is written as:

$E = \{2, 4, 6\} \Rightarrow n(E) = 3$

Mutually Exclusive Events: If a sample space $S = \{1, 3, 5, 7, 9\}$ and an event $A = \{1, 3, 5\}$ and another event $B = \{9\}$, then *A* and *B* are disjoint sets and they are said to be **mutually exclusive events**. In tossing a coin, the sample space $S = \{H, T\}$. Now, if event $A = \{H\}$ and event $B = \{T\}$, then *A* and *B* are mutually exclusive events.

Equally Likely Events: We know that if a fair coin is tossed, the chance of **head** appearing

on the top is the same as that of the **tail**. We say that these two events are **equally likely**. Similarly, if a die, which is a perfect unloaded cube is rolled, then the face containing 2 dots is as likely to be on the top as the face containing 5 dots. The same will be the case with any other pair of faces. In general, if two events A and B occur in an experiment, then ***A* and *B* are said to be equally likely events if each one of them has equal number of chances of occurrence.**

The following definition of Probability was given by a French Mathematician, P.S. Laplace (1749 - 1827) and it has been accepted as a standard definition by the mathematicians all over the world:

If a random experiment produces *m* different but equally likely out-comes and *n* outcomes out of them are favourable to the occurrence of the event *E*, then the probability of the occurrence of the event *E* is denoted by *P*(*E*) such that

$$P(E) = \frac{n}{m} = \frac{n(E)}{n(S)} = \frac{\text{no.of ways in which event occurs}}{\text{no.the elements of the sample space}}$$

Since the number of outcomes in an event is less than or equal to the number of outcomes in the sample space, the probability of an event must be a number between 0 and 1.

That is, $0 \leq P(E) \leq 1$

i) If P{E) = 0, event E cannot occur and E is called an impossible event.

ii) If P(E) = 1, event E is sure to occur and E is called a certain event.

Example 1: A die is rolled. What is the probability that the dots on the top are greater than 4?

Solution: $S = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6$

The event E that the dots on the top are greater than 4 = {5, 6 }

$\Rightarrow n(E) = 2 \qquad \therefore P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} \quad \frac{1}{3}$

Example 2: What is the probability that a slip of numbers divisible by 4 are picked from the slips bearing numbers 1, 2, 3, ...,10

Solution: $S = \{1, 2, 3, ..., 10\} \Rightarrow n(S) = 10$

Let *E* be the event of picking slip with number divisible by 4.

$E = \{4,8\} \Rightarrow n(E) = 2$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{2}{10} = \frac{1}{5}$$

7.4.1 Probability that an Event does not Occur

If a sample space S is such that $n(S) = N$ and out of the N equally likely events an event E occurs R times, then, evidently, E does not occur $N - R$ times.

The non-occurrence of the event E is denoted as $\overline{E}$.

Now $P(E) = \frac{n(E)}{n(S)} = \frac{R}{N}$

and $P(\overline{E}) = \frac{n(\overline{E})}{n(S)} = \frac{N-R}{N} = \frac{N}{N} - \frac{R}{N} = 1 - \frac{R}{N}$

$\therefore \quad P(\overline{E}) = 1 - p(E).$

Exercise 7.5

For the following experiments, find the probability in each case:

1. Experiment:
From a box containing orange-flavoured sweets, Bilal takes out one sweet without looking.
Events Happening:
i) the sweet is orange-flavoured
ii) the sweet is lemon-flavoured.

2. Experiment:
Pakistan and India play a cricket match. The result is:
Events Happening: i) Pakistan wins ii) India does not lose.

3. Experiment:
There are 5 green and 3 red balls in a box, one ball is taken out.
Events Happening: i) the ball is green ii) the ball is red.

4. Experiment:
A fair coin is tossed three times. It shows
Events Happening: i) One tail ii) atleast one head.

5. Experiment:
A die is rolled. The top shows
Events Happening: i) 3 or 4 dots ii) dots less than 5.

6. Experiment:
From a box containing slips numbered 1,2, 3, ..., 5 one slip is picked up
Events Happening:
i) the number on the slip is a prime number
ii) the number on the slip is a multiple of 3.

7. Experiment:
Two die, one red and the other blue, are rolled simultaneously. The numbers of dots on the tops are added. The total of the two scores is:
Events Happening: i) 5 ii) 7 iii) 11.

8. Experiment:
A bag contains 40 balls out of which 5 are green, 15 are black and the remaining are yellow. A ball is taken out of the bag.
Events Happening:
i) The ball is black ii) The ball is green iii) The ball is not green.

9. Experiment:
One chit out of 30 containing the names of 30 students of a class of 18 boys and 12 girls is taken out at random, for nomination as the monitor of the class.
Events Happening:
i) the monitor is a boy ii) the monitor is a girl.

10. Experiment:
A coin is tossed four times. The tops show
Events Happening:
i) all heads ii) 2 heads and 2 tails.

7.4.2 Estimating Probability and Tally Marks

We know that $P(E)=\dfrac{n(E)}{n(S)}$, where E is the event and S is the sample space. The fraction showing the probability is very often such that it is better to find its approximate value. The following examples illustrate the necessity of approximation.

Example 1: The table given below shows the result of rolling a die 100 times. Find the probability in which odd numbers occur.

Event	Tally Marks	Frequency
1	卌 卌 卌 卌 卌	25
2	卌 卌 |||	13
3	卌 卌 ||||	14
4	卌 卌 卌 卌 ||||	24
5	卌 |||	8
6	卌 卌 卌 |	16

Solution: Required probability $=\dfrac{25+14+8}{100}=\dfrac{47}{100}=\dfrac{1}{2}$ (approx.)

Note: In the above experiment, we have written the probability = $\frac{1}{2}$ (approx.)

It may be remembered that the greater the number of trials, the more accurate is the estimate of the probability.

Example 2: The number of rainy days in Murree during the month of July for the past ten years are: 20, 20, 22, 22, 23, 21, 24, 20, 22, 21

Estimate the probability of the rain falling on a particular day of July. Hence find the number of days in which picnic programme can be made by a group of students who wish to spend 20 days in Murree.

Solution: Let E be the event that rain falls on a particular day of a July.

$$P(E)=\frac{20+20+22+22+23+21+24+20+22+21}{31\times 10}$$

$$=\frac{215}{310}=0.7 \qquad \text{(approx).}$$

Number of days of raining in 20 days of July = 20 × 0.7 = 14

∴ The number of days fit for picnic = 2 0 – 1 4 = 6

Exercise 7.6

1. A fair coin is tossed 30 times, the result of which is tabulated below. Study the table and answer the questions given below the table:

Event	Tally Marks	Frequency
Head	卌 卌 ||||	14
Tail	卌 卌 卌 |	16

i) How many times does 'head' appear?

ii) How many times does 'tail' appear?

iii) Estimate the probability of the appearance of head?

iv) Estimate the probability of the appearance of tail?

2. A die is tossed 100 times. The result is tabulated below. Study the table and answer the questions given below the table:

Event	Tally Marks	Frequency
1	卌 卌 ||||	14
2	卌 卌 卌 ||	17
3	卌 卌 卌 卌	20
4	卌 卌 卌 |||	18
5	卌 卌 卌	15
6	卌 卌 卌 |	16

i) How many times do 3 dots appear?

ii) How many times do 5 dots appear?

iii) How many times does an even number of dots appear?

iv) How many times does a prime number of dots appear?
v) Find the probability of each one of the above cases.

3. The eggs supplied by a poultry farm during a week broke during transit as follows:

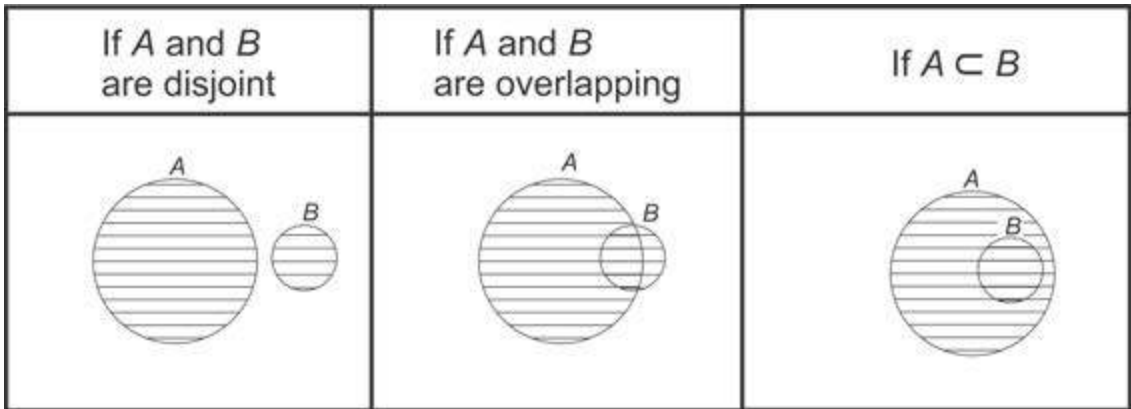
$$1\%,\ 2\%,\ 1\frac{1}{2}\%,\ \frac{1}{2}\%,\ 1\%,\ 2\%,\ 1\%$$

Find the probability of the eggs that broke in a day. Calculate the number of eggs that will be broken in transiting the following number of eggs:

i) 7,000 ii) 8,400 iii) 10,500

7.4.3 Addition of Probabilities

We have learnt in chapter 1, that if *A* and *B* are two sets, then the shaded parts in the following diagram represent $A \cup B$.

If *A* and *B* are disjoint	If *A* and *B* are overlapping	If $A \subset B$
A B	A B	A B

The above diagrams help us in understanding the formulas about the sum of two probabilities.

We know that:

P(*E*) is the probability of the occurrence of an event *E*.

If *A* and *B* are two events, then

P(*A*) = the probability of the occurrence of event *A*;
P(*B*) = the probability of the occurrence of event B;
P(*A*∪*B*) = the probability of the occurrence of $A \cup B$;
P(*A* ∩ *B*) = the probability of the occurrence of $A \cap B$;

The formulas for the addition of probabilities are:

i) $P(A\cup B) = P(A) + P(B)$, when *A* and *B* are disjoint

ii) $P(A\cup B) = P(A) + P(B) - P(A \cap B)$
when *A* and *B* are overlapping or $B \subseteq A$.

Let us now learn the application of these formulas in solving problems involving the addition of two probabilities.

Example 1: There are 20 chits marked 1, 2, 3,....,20 in a bag. Find the probability of picking a chit, the number written on which is a multiple of 4 or a multiple of 7.

Solution: Here $S = \{1, 2, 3, ..., 20\} \Rightarrow n(S) = 20$

Let A be the event of getting multiples of 4.

$\therefore\ A = \{4, 8, 12, 16, 20\} \Rightarrow n(A) = 5$

$$\therefore P(A) = \frac{5}{20} = \frac{1}{4}$$

Let *B* be the event of getting multiples of 7

$\therefore\ B = \{7, 14\} \Rightarrow n(B) = 2$

$$\therefore P(B) = \frac{2}{20} = \frac{1}{10}$$

As *A* and *B* are disjoint sets

$$\therefore\ P(A\cup B) = p(A) + p(B) = \frac{1}{4} + \frac{1}{10} = \frac{7}{20}$$

Example 2: A die is thrown. Find the probability that the dots on the top are prime numbers or odd numbers.

Solution: Here $S = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6$

Let A = Set of prime numbers = $\{2,3,5\} \Rightarrow n(A) = 3$

Let B = Set of odd numbers = $\{1,3,5\} \Rightarrow n(B) = 3$

$\therefore A \cap B = \{2, 3, 5\} \cap \{1, 3, 5\} = \{3, 5\} \Rightarrow n(A \cap B) = 2$

Now $P(A) = \frac{3}{6} = \frac{1}{2}$, $P(B) = \frac{3}{6} = \frac{1}{2}$, $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$

Since A and B are overlapping sets.

$$P(A\cup B) = P(A)+P(B)-P(A\cap B)$$

$$=\frac{1}{2}+\frac{1}{2}-\frac{1}{3}=\frac{2}{3}$$

Exercise 7.7

1. If sample space= {1, 2, 3 , 9 } , Event A = {2, 4, 6, 8} and Event B = {1, 3, 5}, find $P(A\cup B)$.
2. A box contains 10 red, 30 white and 20 black marbles. A marble is drawn at random. Find the probability that it is either red or white.
3. A natural number is chosen out of the first fifty natural numbers. What is the probability that the chosen number is a multiple of 3 or of 5?
4. A card is drawn from a deck of 52 playing cards. What is the probability that it is a diamond card or an ace?
5. A die is thrown twice. What is the probability that the sum of the number of dots shown is 3 or 11?
6. Two dice are thrown. What is the probability that the sum of the numbers of dots appearing on them is 4 or 6 ?
7. Two dice are thrown simultaneously. If the event A is that the sum of the numbers of dots shown is an odd number and the event B is that the number of dots shown on at least one die is 3. Find $P(A\cup B)$.
8. There are 10 girls and 20 boys in a class. Half of the boys and half of the girls have blue eyes. Find the probability that one student chosen as monitor is either a girl or has blue eyes.

7.4.4 Multiplication of Probabilities

We can multiply probabilities of **dependent** as well as **independent** events. But, in this section, we shall find the multiplication of independent events only. Before learning the formula of the multiplication of the probabilities of independent events, it is necessary to understand that what is meant by independent events.

Two events A and B are said to be independent; if the occurrence of any one of

them does not influence the occurrence of the other event. In other words, regardless of whether event A has or has not occurred, if the probability of the event B remains the same, then A and B are independent events.

Suppose a bag contains 12 balls. If 4 balls are drawn from it twice in such a way that:

i) the balls of the first draw are not replaced before the second draw;

ii) the balls of the first draw are replaced before the second draw.

In the case (i), the second draw will be out of (12 – 4 = 8) balls which means that the out-comes of the second draw will depend upon the events of the first draw and the two events will not be independent. However, in case (ii), the number of balls in the bag will be the same for the second draw as has been the case at the time of first draw i.e. the first draw will not influence the probability of the event of second draw. So the two events in this case will be independent.

Theorem: If A and B are two independent events, the probability that both of them occur is equal to the probability of the occurrence of A multiplied by the probability of the occurrence of B. Symbolically, it is denoted as:

$$P(A\cap B) = P(A).P(B)$$

Proof: Let event A belong to the sample space S_1 such that

$$n(S_1) = n_1 \quad \text{and} \quad n(A) = m_1 \quad \Rightarrow P(A)=\frac{m_1}{n_1}$$

Let event B belong to the sample space S_2 such that

$$n(S_2) = n_2 \quad \text{and} \quad n(B) = m_2 \quad \Rightarrow p(B)=\frac{m_2}{n_2}$$

$\because$ A and B are independent events

$\therefore$ Total number of combined outcomes of A and B = $n_1 n_2$

and total number of favourable outcomes = $m_1 m_2$

$$\therefore \quad P(A\cap B) = \frac{m_1 m_2}{n_1 n_2} = \frac{m_1}{n_1}.\frac{m_2}{n_2} = P(A).P(B)$$

Note: The above proof of the formula holds good even if the sample spaces of A and B are the same. The formula $P(A\cap B) = P(A).P(B)$ can be generalized as:

$$P(A_1\cap A_2\cap A_3\cap.....\cap A_n) = P(A_1).P(A_2).P(A_3)........P(A_n)$$

where $A_1, A_2, A_3, ..., A_n$ are independent events.

Example 1 The probabilities that a man and his wife will be alive in the next 20 years are 0.8 and 0.75 respectively. Find the probability that both of them will be alive in the next 20 years.

Solution: If $P(A)$ is the probability that the man will be alive in 20 years and $P(B)$ is the probability that his wife be alive in 20 years.

$\because$ The two events are independent:

$\therefore$ $P(A) = 0.8$ $\quad P(B) = 0.75$

The probability that both man and wife will be alive in 20 years is given by:

$$P(A\cap B) = 0.8\times 0.75 = 0.6$$

Example 2: Two dice are thrown. E_1 is the event that the sum of their dots is an odd number and E_2 is the event that 1 is the dot on the top of the first die. Show that $P(E_1\cap E_2) = P(E_1).P(E_2)$

Solution: E_1 = {(1,2), (1 ,4), (1, 6), (2, 3), (2, 5), (3,4), (3 ,6), (4, 3) (4, 5) (5, 6), (2, 1), (4, 1), (6,1), (3, 2), (5, 2), (6, 3), (5, 4), (6, 5)}

$\Rightarrow$ $n(E_1) = 18$

E_2 = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)}

$\Rightarrow$ $n(E_2) = 6$

$\because$ $n(S) = 6\times 6 = 36$

$\therefore$ $P(E_1) = \frac{18}{36} = \frac{1}{2}$ and $P(E_2) = \frac{6}{36} = \frac{1}{6}$

$\because$ E_1 and E_2 are independent

$\therefore$ $P(E_1).P(E_2) = \frac{1}{2}.\frac{1}{6} = \frac{1}{12}$

Now $E_1\cap E_2$ = {(1,2), (1,4), (1,6)}

$\Rightarrow$ $n(E_1\cap E_2) = 3$

$\therefore$ $P(E_1\cap E_2) = \frac{3}{36} = \frac{1}{12}$

Hence $P(E_1\cap E_2) = P(E_1).P(E_2)$

Exercise 7.8

1. The probability that a person A will be alive 15 years hence is $\frac{5}{7}$ and the probability that another person B will be alive 15 years hence is $\frac{7}{9}$. Find the probability that both will be alive 15 years hence.
2. A die is rolled twice: Event E_1 is the appearance of even number of dots and event E_2 is the appearance of more than 4 dots. Prove that: $P(E_1\cap E_2) = P(E_1).P(E_2)$
3. Determine the probability of getting 2 heads in two successive tosses of a balanced coin.
4. Two coins are tossed twice each. Find the probability that the head appears on the first toss and the same faces appear in the two tosses.
5. Two cards are drawn from a deck of 52 playing cards. If one card is drawn and replaced before drawing the second card, find the probability that both the cards are aces.
6. Two cards from a deck of 52 playing cards are drawn in such a way that the card is replaced after the first draw. Find the probabilities in the following cases:
   i) first card is king and the second is a queen.
   ii) both the cards are faced cards i.e. king, queen, jack.
7. Two dice are thrown twice. What is probability that sum of the dots shown in the first throw is 7 and that of the second throw is 11 ?
8. Find the probability that the sum of dots appearing in two successive throws of two dice is every time 7.

9. A fair die is thrown twice. Find the probability that a prime number of dots appear in the first throw and the number of dots in the second throw is less than 5.

10. A bag contains 8 red, 5 white and 7 black balls, 3 balls are drawn from the bag. What is the probability that the first ball is red, the second ball is white and the third ball is black, when every time the ball is replaced?

HINT: $\left(\frac{8}{20}\right)\left(\frac{5}{20}\right)\left(\frac{7}{20}\right)$ is the probability.

CHAPTER

8

Mathematical Inductions and Binomial Theorem

8.1 Introduction

Francesco Mourolico (1494-1575) devised the method of induction and applied this device first to prove that the sum of the first n odd positive integers equals n^2. He presented many properties of integers and proved some of these properties using the method of *mathematical induction.*

We are aware of the fact that even one exception or case to a mathematical formula is enough to prove it to be false. Such a case or exception which fails the mathematical formula or statement is called a counter example.

The validity of a formula or statement depending on a variable belonging to a certain set is established if it is true for each element of the set under consideration.

For example, we consider the statement $S(n) = n^2 - n + 41$ is a prime number for every natural number n. The values of the expression $n^2 - n + 41$ for some first natural numbers are given in the table as shown below:

n	1	2	3	4	5	6	7	8	9	10	11
S(n)	41	43	47	53	61	71	83	91	113	131	151

From the table, it appears that the statement $S(n)$ has enough chance of being true. If we go on trying for the next natural numbers, we find $n = 41$as a counter example which fails the claim of the above statement. So we conclude that to derive a general formula without proof from some special cases is not a wise step. This example was discovered by Euler (1707-1783).

Now we consider another example and try to formulate the result. Our task is to find the sum of the first n odd natural numbers. We write first few sums to see the pattern of sums.

n (The number of terms) Sum

1 ---1= 1^2

2 --1+3= 4 = 2^2

3 ---1+3+5= 9=3^2

4 --1+3+5+7=16=4^2

5 ---1+3+5+7+9=25 = 5^2

6 ---1+3+5+7+9+11=36= 6^2

The sequence of sums is $(1)^2$, $(2)^2$, $(3)^2$, $(4)^2$, ...

We see that each sum is the square of the number of terms in the sum. So the following statement seems to be true.

For each natural number n,

$1 + 3 + 5 + \dots + (2n - 1) = n^2$ . . . . (i) $(\because$ nth term$=1+(n-1)2)$

But it is not possible to verify the statement (i) for each positive integer n, because it involves infinitely many calculations which never end.

The method of mathematical induction is used to avoid such situations.Usually it is used to prove the statements or formulae relating to the set {1,2,3,...} but in some cases, it is also used to prove the statements relating to the set (0 ,1,2 ,3,...}.

8.2 Principle of Mathematical Induction

The principle of mathematical induction is stated as follows:

If a proposition or statement $S(n)$ for each positive integer n is such that

1) $S(1)$ is true i.e., $S(n)$ is true for $n = 1$ and

2) $S(k + 1)$ is true whenever $S(k)$ is true for any positive integer k, then $S(n)$ is true for all positive integers.

Procedure:

1. Substituting $n = 1$, show that the statement is true for $n = 1$.
2. Assuming that the statement is true for any positive integer k, then show that it is true for the next higher integer. For the second condition, one of the following two methods can be used:

M_1 Starting with one side of $S(k+1)$, its other side is derived by using $S(k)$.

M_2 $S(k + 1)$ is established by performing algebraic operations on $S(k)$.

Example 1: Use mathematical induction to prove that $3+6+9+\dots+3n=\frac{3n(n+1)}{2}$ for every positive integer n.

Solution: Let $S(n)$ be the given statement, that is,

$$S(n): \quad 3+6+9....+3n=\frac{3n(n+1)}{2} \qquad \text{(i)}$$

1. When n = 1, S(1) becomes

$$S(1): 3=\frac{3(1)(1+1)}{2}=3$$

Thus S(1) is true i.e., condition (1) is satisfied.

2. Let us assume that $S(n)$ is true for any $n = k \in N$, that is,

$$3+6+9....+3k=\frac{3k(k+1)}{2} \qquad \text{(A)}$$

The statement for $n = k+1$ becomes

$$3+6+9....+3k+3(k+1)=\frac{3k(k+1)\left[(k+1)+1\right]}{2}$$
$$=\frac{3(k+1)(k+2)}{2} \qquad \text{(B)}$$

Adding 3(k+ 1) on both the sides of (A) gives

$$3+6+9+....+3k+3(k+1)=\frac{3k(k+1)}{2}+3(k+1)$$
$$=3(k+1)(\frac{k}{2}+1)$$
$$=\frac{3(k+1)(k+2)}{2}$$

Thus $S(k + 1)$ is true if $S(k)$ is true, so the condition (2) is satisfied.

Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

Example 2: Use mathematical induction to prove that for any positive integer n,

$$1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$$

Solution: Let $S(n)$ be the given statement,

$$S(n): \quad 1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$$

1. If n = 1, S (1) becomes

$$S(1):(1)^2=\frac{1(1+1)(2\times1+1)}{6}=\frac{1\times2\times3}{6}=1$$

Thus S(1) is true, i.e., condition (1) is satisfied.

2. Let us assume that $S(k)$ is true for any $k \in N$, that is,

$$1^2+2^2+3^2+....+k^2=\frac{k(k+1)(2k+1)}{2} \qquad \text{(A)}$$

$$S(k+1):\ 1^2+2^2+3^2+....+k^2+(k+1)^2=\frac{(k+1)(\overline{k+1}+1)(2\overline{k+1}+1)}{6}$$
$$=\frac{(k+1)(k+2)(2k+3)}{6} \qquad \text{(B)}$$

Adding $(k+1)^2$ to both the sides of equation (A), we have

$$1^2+2^2+3^2+....+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$
$$=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$
$$=\frac{(k+1)(2k^2+k+6k+6)}{6}$$
$$=\frac{(k+1)(2k^2+7k+6)}{6}$$
$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

Thus the condition (2) is satisfied. Since both the conditions are satisfied, therefore, by mathematical induction, the given statement holds for all positive integers.

Example 3: Show that $\frac{n^3+2n}{3}$ represents an integer $\forall n \in N$.

Solution: Let $S(n)=\frac{n^3+2n}{3}$

1. When n = 1, S(1) becomes

$$S(1)=\frac{1^3+2(1)}{3}=\frac{3}{3}=1\in Z$$

2. Let us assume that $S(n)$ is ture for any $n = k \in W$, that is,

$$S(k)=\frac{k^3+2k}{3}$$ represents an integer.

Now we want to show that $S(k+1)$ is also an integer. For $n=k+1$, the statement becomes

$$S(k+1)=\frac{(k+1)^3+2(k+1)}{3}$$
$$=\frac{k^3+3k^2+3k+1+2k+2}{3}=\frac{(k^3+2k)+(3k^2+3k+3)}{3}$$
$$=\frac{(k^3+2k)+3(k^2+k+1)}{3}$$
$$=\frac{k^3+2k}{3}+(k^2+k+1)$$

As $\frac{k^3+2k}{3}$ is an integer by assumption and we know that $(k^2 + k + 1)$is an integer as $k \in W$.

$S(k + 1)$ being sum of integers is an integer, thus the condition (2) is satisfied.

Since both the conditions are satisfied, therefore, we conclude by mathematical induction that $\frac{n^3+2n}{3}$ represents an integer for all positive integral values of n.

Example 4: Use mathematical induction to prove that

$3+3.5+3.5^2+....+3.5^n=\frac{3(5^{n+1}-1)}{4}$ whenever n is non-negative integer.

Solution: Let $S(n)$ be the given statement, that is,

$$S(n):\quad 3+3.5+3.5^2+....+3.5^n=\frac{3(5^{n+1}-1)}{4}$$

The dot (.)between two number, stands, for multipication symbol.

1. For n= 0, S(0) becomes $S(0):3.5^0=\frac{3(5^{0+1}-1)}{4}$ or $3=\frac{3(5-1)}{4}=3$

Thus S(0) is true i.e., conditions (1) is satisfied.

2. Let us assume that $S(k)$ is true for any $k \in W$, that is,

$$S(k):3+3.5+3.5^2+....+3.5^k=\frac{3(5^{k+1}-1)}{4} \quad \text{(A)}$$

Here $S(k+1)$ becomes

$$S(k+1):3+3.5+3.5^2+....+3.5^k+3.5^{k+1}=\frac{3(5^{(k+1)+1}-1)}{4}$$
$$=\frac{3(5^{k+2}-1)}{4} \quad \text{(B)}$$

Adding 3.5^{k+1} on both sides of (A), we get

$$3+3.5+3.5^2+....+3.5^k+3.5^{k+1}=\frac{3(5^{k+1}-1)}{4}+3.5^{k+1}$$
$$=\frac{3(5^{k+1}-1+4.5^{k+1})}{4}$$
$$=\frac{3[5^{k+1}(1+4)-1]}{4}$$
$$=\frac{3(5^{k+2}-1)}{4}$$

This shows that $S(k + 1)$ is true when $S(k)$ is true. Since both the conditions are satisfied, therefore, by the principle of mathematical induction, $S(n)$ in true for each $n \in W$.

Care should be taken while applying this method. Both the conditions (1) and (2) of the principle of mathematical induction are essential. The condition (1) gives us a starting point but the condition (2) enables us to proceed from one positive integer to the next. In the condition (2) we do not prove that $S(k+1)$ is true but prove only that if $S(k)$is true, then $S(k+1)$ is true. We can say that any proposition or statement for which only one condition is satisfied, will not be truefor all n belonging to the set of positive integers.

For example, we consider the statement that 3^n is an even integer for any positive integer n. Let $S(n)$ be the given statement.

Assume that $S(k)$ is true, that is, 3^k in an even integer for $n = k$. When 3^k is even, then $3^k + 3^k + 3^k$ is even which implies that $3^k .3= 3^{k+1}$ is even.

This shows that $S(k+1)$ will be true when $S(k)$ is true. But 3^1 is not an even integer which reflects that the first condition does not hold. Thus our supposition is false.

Note:- There is no integer n for which 3^n is even.

Sometimes, we wish to prove formulae or statements which are true for all integers n greater than or equal to some integer i, where $i \neq 1$. In such cases, $S(1)$ is replaced by $S(i)$ and the condition (2) remains the same. To tackle such situations, we use the principle of extended mathematical induction which is stated as below:

8.3 Principle of Extended Mathematical Induction

Let i be an integer. If a formula or statement $S(n)$ for $n \geq i$ is such that

1) $S(i)$ is true and
2) $S(k+1)$ is true whenever $S(k)$ is true for any integer $n \geq i$.

Then $S(n)$ is true for all integers $n \geq i$.

Example 5: Show that $1 + 3 + 5 + \dots (2n+5) = (n+3)^2$ for integral values of $n \geq -2$.

Solution:

1. Let $S(n)$ be the given statement, then for $n = -2$, $S(-2)$ becomes, $2(-2)+5=(-2+3)^2$, i.e., $1 = (1)^2$ which is true.
   Thus $S(-2)$ is true i.e., the condition (1) is satisfied
2. Let the equation be true for any $n = k \in Z$, $k \geq -2$, so that
   $1+3+5+\dots+(2k+5)= (k+3)^2$ (A)

$S(k+1): 1+3+5+\dots+(2k+5)+(2\overline{k+1}+5)=(\overline{k+1}+3)^2=(k+4)^2$ (B)

Adding $(2\overline{k+1}+5)=(2k+7)$ on both sides of equation (A) we get,

$$\begin{aligned} 1+3+5+\dots+(2k+5)+(2k+7) &= (k+3)^2+(2k+7) \\ &= k^2+6k+9+2k+7 \\ &= k^2+8k+16 \\ &= (k+4)^2 \end{aligned}$$

Thus the condition (2) is satisfied. As both the conditions are satisfied, so we conclude that the equation is true for all integers $n \geq -2$.

Example 6: Show that the inequality $4^n > 3^n + 4$ is true, for integral values of $n \geq 2$.

Solution: Let $S(n)$ represents the given statement i.e., $S(n)$: $4^n > 3^n + 4$ for integral values of $n \geq 2$.

1. For $n = 2$, $S(2)$ becomes
   $S(2)$: $4^2 > 3^2 + 4$, i.e., $16 > 13$ which is true.
   Thus $S(2)$ is true, i.e., the first condition is satisfied.
2. Let the statement be true for any $n = k (\geq 2) \in Z$, that is
   $4^k > 3^k + 4$ (A)
   Multiplying both sides of inequality (A) by 4, we get
   or $4.4^k > 4(3^k + 4)$
   or $4^{k+1} > (3+1)3^k + 16$
   or $4^{k+1} > 3^{k+1} + 4 + 3^k + 12$
   or $4^{k+1} > 3^{k+1} + 4$ $(\because 3^k + 12 > 0)$ (B)
   The inequality (B), satisfies the condition (2).

Since both the conditions are satisfied, therefore, by the principle of extended mathematical induction, the given inequality is true for all integers $n \geq 2$.

Exercise 8.1

Use mathematical induction to prove the following formulae for every positive integer n.

1. $1+5+9+\dots+(4n-3)=n(2n-1)$

2. $1+3+5+....+(2n-1)=n^2$

3. $1+4+7+....+(3n-2)=\frac{n(3n-1)}{2}$

4. $1+2+4+....+2^{n-1}=2^n-1$

5. $1+\frac{1}{2}+\frac{1}{4}+....+\frac{1}{2^{n-1}}=2\left[1-\frac{1}{2^n}\right]$

6. $2+4+6+....+2n=n(n+1)$

7. $2+6+18+....+2\times 3^{n-1}=3^n-1$

8. $1\times 3+2\times 5+3\times 7+....+n\times(2n+1)=\frac{n(n+1)(4n+5)}{6}$

9. $1\times 2+2\times 3+3\times 4+....+n\times(n+1)=\frac{n(n+1)(n+2)}{3}$

10. $1\times 2+3\times 4+5\times 6+....+(2n-1)\times 2n=\frac{n(n+1)(4n-1)}{3}$

11. $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+....+\frac{1}{n(n+1)}=1 \quad \frac{1}{n+1}$

12. $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

13. $\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+....+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)}$

14. $r+r^2+r^3+....+r^n=\frac{r(1-r^n)}{1-r}, \qquad (r \neq 1)$

15. $a+(a+d)+(a+2d)+....+[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d]$

16. $1.\underline{|1}+2\underline{|2}+3\underline{|3}+....+n\underline{|n}=\underline{|n+1}-1$

17. $a_n=a_1+(n-1)d$ when, $a_1, a_1+d, a_1+2d,$ .... form an A.P.

18. $a_n=a_1r^{n-1}$ when $a_1, a_1r, a_1r^2,$... form a G.P.

19. $1^2+3^2+5^2+....+(2n-1)^2=\frac{n(4n^2-1)}{3}$

20. $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+....+\binom{n+2}{3}=\binom{n+3}{4}$

21. Prove by mathematical induction that for all positive integral values of n

i) n^2+n is divisible by 2. ii) 5^n – 2” is divisible by 3.

iii) 5^n-1 is divisible by 4. iv) $8 \times 10^n - 2$ is divisible by 6.

v) n^3-n is divisible by 6.

22. $\frac{1}{3}+\frac{1}{3^2}+....+\frac{1}{3^n}=\frac{1}{2}\left[1-\frac{1}{3^n}\right]$

23. $1^2-2^2+3^2-4^2+....+(-1)^{n-1}.n^2=\frac{(-1)^{n-1}.n(n+1)}{2}$

24. $1^3+3^3+5^3+....+(2n-1)^3=n^2[2n^2-1]$

25. $x+1$ is a factor of $x^{2n}-1; (x\neq -1)$

26. $x-y$ is a factor of x^n-y^n; $(x \neq y)$

27. $x+y$ is a factor of $x^{2n-1}+y^{2n-1}$ $(x \neq -y)$

28. Use mathematical induction to show that
$1+2+2^2+....+2^n=2^{n+1}-1$ for all non-negative integers n.

29. If A and B are square matrices and $AB=BA$, then show by mathematical induction that $AB^n=B^nA$ for any positive integer n.

30. Prove by the Principle of mathematical induction that n^2-1 is divisible by 8 when n is an odd positive integer.

31. Use the principle of mathematical induction to prove that $\ln x^n = n \ln x$ for any integer $n \geq 0$ if x is a positive number. Use the principle of extended mathematical induction to prove that:

32. $n! > 2^n - 1$ for integral values of $n \geq 4$.

33. $n^2 > n + 3$ for integral values of $n \geq 3$.

34. $4^n > 3^n + 2^{n-1}$ for integral values of $n \geq 2$.

35. $3^n < n!$ for integral values of $n > 6$.

36. $n! > n^2$ for integral values of $n \geq 4$.

37. $3 + 5 + 7 + ... + (2n + 5) = (n + 2)(n + 4)$ for integral values of $n \geq -1$.
38. $1 + nx \leq (1 + x)^n$ for $n \geq 2$ and $x > -1$

8.4 Binomial Theorem

An algebraic expression consisting of two terms such as $a + x$, $x - 2y$, $ax + b$ etc., is called a binomial or a binomial expression.

We know by actual multiplication that

$(a + x)^2 = a^2 + 2ax + x^2$ (i)

$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$ (ii)

The right sides of (i) and (ii) are called binomial expansions of the binomial $a + x$ for the indices 2 and 3 respectively.

In general, the rule or formula for expansion of a binomial raised to any positive integral power n is called the binomial theorem for positive integral index n.For any positive integer n,

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + ... + \binom{n}{r-1}a^{n-(r-1)}x^{r-1} \quad \text{(A)}$$

$$+\binom{n}{r}a^{n-r}x^r + ... + \binom{n}{n-1}ax^{n-1} + \binom{n}{n}x^n$$

or briefly

$$(a+x)^n = \sum_{r=0}^{n} \binom{n}{r}a^{n-r}x^r$$

where a and x are real numbers.

The rule of expansion given above is called the binomial theorem and it also holds if a or x is complex.

Now we prove the Binomial theorem for any positive integer n, using the principle of mathematical induction.

Proof: Let $S(n)$ be the statement given above as (A).

1. If $n = 1$, we obtain

$$S(1): (a+x)^1 = \binom{1}{0}a^1 + \binom{1}{1}a^{1-1}x = a + x$$

Thus condition (1) is satisfied.

2. Let us assume that the statement is true for any $n = k \in N$, then

$$(a+x)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}x + \binom{k}{2}a^{k-2}x^2 + ... + \binom{k}{r-1}a^{k-(r-1)}x^{r-1} \quad \binom{k}{r}a^{k-r}x^r$$

$$+ .... + \binom{k}{k}ax^k \quad + \binom{k}{k}x^k \quad \text{(B)}$$

$$S(k+1): (a+k)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k \times x + \binom{k+1}{2}a^{k-1} \times x^2 + ... +$$

$$\binom{k+1}{r-1}a^{k-r+2} \times x^{r-1} + \binom{k+1}{r}a^{k-r+1} \times x^r + ... + \binom{k+1}{k}a \times x^k + \binom{k+1}{k+1}x^{k+1} \quad \text{(C)}$$

Multiplying both sides of equation (B) by $(a+x)$, we have

$$(a+x)(a+x)^k = (a+x)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x + \binom{k}{2}a^{k-2}x^2 + ... + \binom{k}{r-1}a^{k-r+1}x^{r-1}\right.$$

$$\left. + \binom{k}{r}a^{k-r} \quad x^r + ... + \binom{k}{k-1}ax^{k-1} + \binom{k}{k}x^k\right]$$

$$= \left[\binom{k}{0}a^{k+1} + \binom{k}{1}a^kx + \binom{k}{2}a^{k-1}x^2 + ... + \binom{k}{r-1}a^{k-r+2}x^{r-1}\right.$$

$$\left. + \binom{k}{r}a^{k-r+1}x^r + ... + \binom{k}{k-1}a^2x^{k-1} + \binom{k}{k}ax^k\right]$$

$$+\left[\binom{k}{0}a^{k}x+\binom{k}{1}a^{k-1}x^{2}+\binom{k}{2}a^{k-2}\times x^{3}+....+\binom{k}{r-1}a^{k-r+1}\times x^{r}\right.$$

$$\left.+\binom{k}{r}a^{k-r}x^{r+1}+....+\binom{k}{k-1}ax^{k}+\binom{k}{k}x^{k+1}\right]$$

$$=\binom{k}{0}a^{k+1}+\left[\binom{k}{1}+\binom{k}{0}\right]a^{k}x+\left[\binom{k}{2}+\binom{k}{1}\right]a^{k-1}x^{2}+....$$

$$+\left[\binom{k}{r}+\binom{k}{r-1}\right]a^{k-r+1}x^{r}+....+\left[\binom{k}{k}+\binom{k}{k-1}\right]ax^{k}+\binom{k}{k}x^{k+1}$$

As $\binom{k}{0}=\binom{k+1}{0},\binom{k}{k}=\binom{k+1}{k+1}\text{and}\binom{k}{r}+\binom{k}{r-1}=\binom{k+1}{r}\text{for} 1\le r\le k$

$$\therefore (a+x)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^{k}x+\binom{k+1}{2}a^{k-1}x^{2}+....$$

$$+\binom{k+1}{r}a^{k-r+1}x^{r}+....+\binom{k+1}{k}a\ x^{k}+\binom{k+1}{k+1}x^{k+1}$$

We find that if the statement is true of $n = k$, then it is also true for $n = k +1$. Hence we conclude that the statement is true for all positive integral values of n.

Note: $\binom{n}{0},\binom{n}{1},\binom{n}{2},....,\binom{n}{n}$ are called the binomial coefficients.

The following points can be observed in the expansion of $(a + x)^n$

1. **The number of terms in the expansion is one greater than its index.**
2. **The sum of exponents of a and x in each term of the expansion is equal to its index.**
3. **The exponent of a decreases from index to zero.**
4. **The exponent of x increases from zero to index.**
5. **The coefficients of the terms equidistant from beginning and end of the expansion are equal as** $\binom{n}{r}=\binom{n}{n-r}$
6. **The $(r$ + l)th term in the expansion** $\binom{n}{r}a^{n-r}x^{r}$ **and we denote it as** T_{r+1} **i.e.,**

$$T \quad \left(\;\right)a^{n\;\;r}x^{r}$$

As all the terms of the expansion can be got from it by putting r = 0, 1,2..... n, so we call it as the **general term** of the expansion.

Example 1: Expand $\left(\frac{a}{2}-\frac{2}{a}\right)^{6}$ also find its general term.

Solution: $\left(\frac{a}{2}-\frac{2}{a}\right)^{6}=\left(\frac{a}{2}+\left(\frac{-2}{a}\right)\right)^{6}$

$$=\left(\frac{a}{2}\right)^{6}+\binom{6}{1}\left(\frac{a}{2}\right)^{5}\left(-\frac{2}{a}\right)+\binom{6}{2}\left(\frac{a}{2}\right)^{4}+\left(-\frac{2}{a}\right)^{2}+\binom{6}{3}\left(\frac{a}{2}\right)^{3}\left(-\frac{2}{a}\right)^{3}$$

$$+\binom{6}{4}\left(\frac{a}{2}\right)^{2}\left(-\frac{2}{a}\right)^{4}+\binom{6}{5}\left(\frac{a}{2}\right)\left(-\frac{2}{a}\right)^{5}+\left(-\frac{2}{a}\right)^{6}$$

$$=\frac{a^{6}}{64}+6\left(\frac{a^{5}}{32}\right)\left(-\frac{2}{a}\right)+\frac{6.5}{2.1}.\frac{a^{4}}{16}.\frac{4}{a^{2}}.+\frac{6.5.4}{3.2.1}.\frac{a^{3}}{8}.\left(-\frac{8}{a^{3}}\right)+\frac{6.5}{2.1}.\frac{a^{2}}{4}.\frac{16}{a^{4}}$$

$$+6.\frac{a}{2}\left(\frac{-32}{a^5}\right)+\frac{64}{a^6}$$

$$=\frac{a^6}{64}-\frac{3}{8}a^4+\frac{15}{4}a^2-20+\frac{60}{a^2}-\frac{96}{a^4}+\frac{64}{a^6}$$

T_{r+1}, the general term is given by

$$T_{r+1}=\binom{6}{r}\left(\frac{a}{2}\right)^{6-r}\left(-\frac{2}{a}\right)^{r}=\binom{6}{r}\frac{a^{6-r}}{2^{6-r}}(-1)^r\frac{2^r}{a^r}$$

$$=(-1)^r\binom{6}{r}\frac{a^{6-r}.a^{-r}}{2^{6-r}.2^{-r}} = (-1)^r.\binom{6}{r}\frac{a^{6-2r}}{2^{6-2r}} = (-1)^r\binom{6}{r}\left(\frac{a}{2}\right)^{6-2r}$$

Example 2: Evaluate $(9.9)^5$

Solution: $(9.9)^5 = (10-.1)^5$

$= (10)^5 + 5 \times (10)4 \times (-.1) + 10(10)^3 \times (-.1)^2 + 10(10)2 \times (-.1)^3 + 5(10)(-.1)^4 + (-.1)^5$

= 100000 – (.5) (10000) + (10000X.01) + 1000(–.001) + 50 (.0001) –.00001

= 100000 – 5000 +100 –1 + .005 – .00001

= 100100.005 – 5001.00001

= 95099.00499

Example 3: Find the specified term in the expansion of $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$;

i) the term involving x^5 ii) the fifth term

iii) the sixth term from the end. iv) coefficient of term involving x^{-1}

Solution:

i) Let T_{r+1} be the term involving x^5 in the expansion of $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$, then

$$T_{r+1}=\binom{11}{r}\left(\frac{3}{2}x\right)^{11-r}\left(-\frac{1}{3x}\right)^{r}=\binom{11}{r}\frac{3^{11-r}}{2^{11-r}}x^{11-r}.(-1)^r.3^{-r}.x^{-r}$$

$$=(-1)^r.\binom{11}{r}\frac{3^{11-2r}}{2^{11-r}}.x^{11-2r}$$

As this term involves x^5 , so the exponent of x is 5, that is,

$$11-2r=5$$

or $-2r=5-11 \Rightarrow r=3$

Thus T_4 involves x^5

$$\therefore\ T_4 = (-1)^3.\binom{11}{3}\frac{3^{11-6}}{2^{11-3}}.x^{11-6} = (-1).\frac{11.10.9}{3.2.1}.\frac{3^5}{2^8}x^5$$

$$= \frac{165\times 243}{256}x^5 \quad - \quad \frac{40095}{256}x^5$$

ii) Putting **r** = 4 in T_{r+1}, we get T_5

$$\therefore\ T_5=(-1)^4\binom{11}{4}\frac{3^{11-8}}{2^{11-4}}.x^{11-8} = \frac{11.10.9.8}{4.3.2.1}.\frac{3^3}{2^7}.x^3$$

$$= \frac{11\times10\times3}{1}.\frac{27}{128}x^3 \quad \frac{165\times 27}{64}x^3$$

$$=\frac{4455}{64}x^3$$

iii) The 6th term from the end term will have (11 + 1) – 6 i.e., 6 terms before it,

∴ It will be (6 + 1) th term i.e.. the 7th term of the expansion.

Thus $T_7=(-1)^6\binom{11}{6}\frac{3^{11-12}}{2^{11-6}}x^{11-12} \quad \frac{11.10.9.8.7}{5.4.3.2.1}\frac{3^{-1}}{2^5}.x^{-1}$

$$=\frac{11\times6\times7}{1}.\frac{1}{3\times32}.\frac{1}{x} \quad \frac{77}{16x}$$

iv) $\frac{77}{16}$ is the coefficient of the term involving x^{-1}

8.3.1 The Middle Term in the Expansion of $(a + x)^n$

In the expansion of $(a + x)^n$, the total number of terms is $n + 1$

Case I: (n is even)

If n is even then $n+1$ is odd, so $\left(\frac{n+1}{2}\right)$th term will be the only one middle term in the expansion.

Case II: (n is odd)

If n is odd then $n + 1$ is even so $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms of the expansion will be the two middle terms.

Example 4: Find the following in the expansion of $\left(\frac{x}{2}+\frac{2}{x^2}\right)^{12}$;

i) the term independent of x. ii) the middle term

Solution: i) Let T_{r+1} be the term independent of x in the expansion of

$\left(\frac{x}{2}+\frac{2}{x^2}\right)^{12}$, then

$$T_{r+1} = \binom{12}{r}\left(\frac{x}{2}\right)^{12-r}\left(\frac{2}{x^2}\right)^{r} = \binom{12}{r}\frac{x^{12-r}}{2^{12-r}}.2^r.x^{-2r}$$

$$= \binom{12}{r}2^{2r-12}.x^{12-3r}$$

As the term is independent of x, so exponent of x, will be zero.

That is, $12 - 3r = 0 \Rightarrow r = 4$.

Therefore the required term $T_5 = \binom{12}{4}2^{8-12}.x^{12-12}$

$$= \frac{12\times11\times10\times9}{4\times3\times2\times1}.2^{-4}.x^0$$

$$= \frac{11\times45}{2^4} = \frac{495}{16}$$

ii) In this case, $n = 12$ which is even, so

$\therefore$ $\left(\frac{12}{2}+1\right)$th term is the middle term in the expansion,

i.e., T_7 is the required term.

$$T_7 = \binom{12}{6}\left(\frac{x}{2}\right)^{12-6}.\left(\frac{2}{x^2}\right)^6$$

$$= \binom{12}{6}\frac{x^6}{2^6}.\frac{2^6}{x^{12}} = \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2\times1}.x^{6-12}$$

$$= \frac{12\times11\times7}{x^6} = \frac{924}{x^6}$$

8.3.2 Some Deductions from the binomial expansion of $(a + x)^n$.

We know that

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \ldots$$

$$+ \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n-1}ax^{n-1} + \binom{n}{n}x^n \qquad \text{(I)}$$

(i) If we put a = 1, in (I), then we have;

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{r}x^r + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \quad \text{(II)}$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)(n-2)....(n-r+1)}{r!}x^r + ... + nx^{n-1} + x^n$$

$$\left(\because \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)....(n-r+1)(n-r)!}{r!(n-r)!} = \frac{n(n-1)....(n-r+1)}{r!}\right)$$

ii) Putting a = 1 and replacing x by $-x$, in (I), we get.

$$(1-x)^n = \binom{n}{0} + \binom{n}{1}(-x) + \binom{n}{2}(-x)^2 + \binom{n}{3}(-x)^3 + ... + \binom{n}{n-1}(-x)^{n-1} + \binom{n}{n}(-x)^n$$

$$= \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + ... + (-1)^{n-1}\binom{n}{n-1}x^{n-1} + (-1)^n\binom{n}{n}x^n.... \quad \text{(III)}$$

iii) We can find the sum of the binomial cofficients by putting a = 1and x = 1 in (I).

i.e., $(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} + \binom{n}{n}$

or $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} + \binom{n}{n}$

Thus the sum of coefficients in the binomial expansion equals to 2^n .

iv) Putting a = 1 and x = –1, in (i) we have

$$(1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n}$$

Thus $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = 0$

If n is odd positive integer, then

$$\binom{n}{0} + \binom{n}{2} + ... + \binom{n}{n-1} = \binom{n}{1} + \binom{n}{3} + ... + \binom{n}{n}$$

If n is even positive integer, then

$$\binom{n}{0} + \binom{n}{2} + ... + \binom{n}{n} = \binom{n}{1} + \binom{n}{3} + ... + \binom{n}{n-1}$$

Thus sum of odd coefficients of a binomial expansion equals to the sum of its even coefficients.

Example 5: Show that: $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + ... + n\binom{n}{n} = n.2^{n-1}$

Solution:

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + ... + n\binom{n}{n} = n + 2\frac{n(n-1)}{2!} + 3\frac{n(n-1)(n-2)}{3!} + ... + n.1$$

$$= n.\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + ... + 1\right]$$

$$= n.\left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + ... + \binom{n-1}{n-1}\right]$$

$$= n.2^{n-1}$$

Exercise 8.2

1. Using binomial theorem, expand the following:

i) $(a+2b)^5$ ii) $\left(\frac{x}{2}-\frac{2}{x^2}\right)^6$ iii) $\left(3a-\frac{x}{3a}\right)^4$

iv) $\left(2a-\frac{x}{a}\right)^7$ v) $\left(\frac{x}{2y}+\frac{2y}{x}\right)^8$ vi) $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6$

2. Calculate the following by means of binomial theorem:
i) $(0.97)^3$ ii) $(2.02)^4$ iii) (9.98) iv) $(21)^5$

3. Expand and simplify the following:

i) $\left(a+\sqrt{2}x\right)^4+\left(a-\sqrt{2}x\right)^4$ ii) $\left(2+\sqrt{3}\right)^5+\left(2-\sqrt{3}\right)^5$

iii) $(2+i)^5-(2-i)^5$ iv) $\left(x+\sqrt{x^2-1}\right)^3+\left(x-\sqrt{x^2-1}\right)^3$

4. Expand the following in ascending power of x:
i) $(2+x-x^2)^4$ ii) $(1-x+x^2)^4$ iii) $(1-x-x^2)^4$

5. Expand the following in descending powers of x:

i) $(x^2+x-1)^3$ ii) $\left(x-1-\frac{1}{x}\right)^3$

6. Find the term involving:
i) x^4 in the expansion of $(3-2x)^7$

ii) x^{-2} in the expansion of $\left(x-\frac{2}{x^2}\right)^{13}$

iii) a^4 in the expansion of $\left(\frac{2}{x}-a\right)^9$

iv) y^3 in the expansion of $\left(x-\sqrt{y}\right)^{11}$

7. Find the coefficient of;

i) x^5 in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$

ii) x^n in the expansion of $\left(x^2-\frac{1}{x}\right)^{2n}$

8. Find 6th term in the expansion of $\left(x^2-\frac{3}{2x}\right)^{10}$

9. Find the term independent of x in the following expansions.

i) $\left(x-\frac{2}{x}\right)^{10}$ ii) $\left(\sqrt{x}+\frac{1}{2x^2}\right)^{10}$ iii) $(1+x^2)^3\left(1+\frac{1}{x^2}\right)^4$

10. Determine the middle term in the following expansions:

i) $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$ ii) $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$ iii) $\left(2x-\frac{1}{2x}\right)^{2m+1}$

11. Find $(2n+1)$ th term from the end in the expansion of $\left(x-\frac{1}{2x}\right)^{3n}$

12. Show that the middle term of $(1+x)^{2n}$ is $=\frac{1.3.5....(2n-1)}{n!}2^n x^n$

13. Show that: $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+....+\binom{n}{n-1}=2^{n-1}$

14. Show that: $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+....+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$

8.4 The Binomial Theorem when the index *n* is a negative integer or a fraction.

When n is a negative integer or a fraction, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$+\frac{n(n-1)(n-2)\ldots.(n-r+1)}{r!}x^r + \ldots..$$

provided $|x|<1$.

The series of the type

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

is called the binomial series.

Note (1): The proof of this theorem is beyond the scope of this book.

(2) Symbols $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}$ etc are meaningless when n is a negative integer or a fraction.

(3) The general term in the expansion is

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots.(n-r+1)}{r!}x^r$$

Example 1: Find the general term in the expansion of $(1+x)^{-3}$ when $|x|<1$

Solution: $T_{r+1} = \dfrac{(-3)(-4)(-5)\ldots.(-3-r+1)}{r!}x^r$

$$= \frac{(-1)^r.3.4.5\ldots.(r+2)}{r!}x^r$$

$$= (-1)^r.\frac{1.2.3.4.5\ldots.(r+2)}{1.2.r!}x^r$$

$$= (-1)^r\frac{r!.(r+1)(r+2)}{2.r!}x^r$$

$$= (-1)^r.\frac{(r+1)(r+2)}{2}x^r$$

Some particular cases of the expansion of $(1+x)^n$, $n<0$.

i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots + (-1)^r x^r + \ldots.$

ii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots. + (-1)^r(r+1)x^r + \ldots$

iii) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots. + (-1)^r\dfrac{(r+1)(r+2)}{2}x^r + \ldots$

iv) $(1-x)^{-1} = 1 + x + x^2 - x^3 + \ldots\ldots + x^r + \ldots.$

v) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots. + (r+1)x^r + \ldots$

vi) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + \dfrac{(r+1)(r+2)}{2}x^r + \ldots$

8.5 Application of the Binomial Theorem

Approximations: We have seen in the particular cases of the expansion of $(1+x)^n$ that the power of x go on increasing in each expansion. Since $|x| < 1$, so

$$|x|^r < |x| \text{ for } r = 2,3,4\ldots$$

This fact shows that terms in each expansion go on decreasing numerically if $|x| < 1$. Thus some initial terms of the binomial series are enough for determining the approximate values of binomial expansions having indices as negative integers or fractions.

Summation of infinite series: The binomial series are conveniently used for summation of infinite series..The series (*whose sum is required*) is compared with

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

to find out the values of n and x. Then the sum is calculated by putting the values of n and x in $(1+x)^n$.

Example 2: Expand $(1-2x)^{1/3}$ to four terms and apply it to evaluate $(.8)^{1/3}$ correct to three places of decimal.

Solution: This expansion is valid only if $|2x|<1$ or $2|x|<1$ or $|x|<\frac{1}{2}$, that is

$$(1-2x)^{1/3}=1+\frac{1}{3}(-2x)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-2x)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-2x)^3+\ldots$$

$$=1-\frac{2}{3}x+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2.1}(4x^2)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3.2.1}(-8x^3)-\ldots$$

$$=1-\frac{2}{3}x-\frac{4}{9}x^2-\frac{1.2.5}{3.3.3}.\frac{1}{3.2.1}(8x^3)-\ldots$$

$$=1-\frac{2}{3}x-\frac{4}{9}x^2-\frac{40}{81}x^3-\ldots$$

Putting $x=.1$ in the above expansion we have

$$(1-2(.1))^{1/3}=1-\frac{2}{3}(.1)-\frac{4}{9}(.1)^2-\frac{40}{81}(.1)^3-\ldots$$

$$=1-\frac{.2}{3}-\frac{.04}{9}-\frac{.04}{81}\ldots \qquad (\because 40\times .001=.04)$$

$$\approx 1-.06666-.00444-.00049=1-.07159\approx .92841$$

Thus $(.8)^{1/3}\approx .928$

Alternative method:

$$(.8)^{1/3}=(1-.2)^{1/3}=1-\frac{.2}{3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-.2)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-.2)^3+\ldots$$

Simplify onward by yourself.

Example 3: Expand $(8-5x)^{-2/3}$ to four terms.

Solution: $(8-5x)^{-2/3}=\left(8\left(1-\frac{5x}{8}\right)\right)^{-\frac{2}{3}}=8^{-\frac{2}{3}}\left(1-\frac{5}{8}x\right)^{-\frac{2}{3}}=(8^{\frac{1}{3}})^{-2}\left(1-\frac{5}{8}x\right)^{-\frac{2}{3}}$

$$=\frac{1}{4}\left(1-\frac{5}{8}x\right)^{-\frac{2}{3}}$$

$$=\frac{1}{4}\left[1+\left(-\frac{2}{3}\right)\left(-\frac{5}{8}x\right)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2!}\left(-\frac{5}{8}x\right)^2+\right.$$

$$\left.\frac{\left(-\frac{2}{3}\right)\left(\frac{-5}{3}\right)\left(-\frac{8}{3}\right)}{3!}\left(-\frac{5}{8}x\right)^3+\ \ldots\right]$$

$$=\frac{1}{4}\left[1+\frac{5}{12}x+\frac{5}{9}\times\frac{25}{64}x^2+\frac{40}{81}\times\frac{125}{8\times 64}x^3+\ldots\right]$$

$$=\frac{1}{4}+\frac{5}{48}x+\frac{125}{2304}x^2+\frac{625}{20736}x^3+\ldots$$

The expansion of $\left(1-\frac{5}{8}x\right)^{-2/3}$ is valid when $\left|\frac{5}{8}x\right|<1$

or $\frac{5}{8}|x|<1 \Rightarrow |x|<\frac{8}{5}$

Example 4: Evaluate $\sqrt[3]{30}$ correct to three places of decimal.

Solution: $\sqrt[3]{30}=(30)^{1/3}=(27+3)^{\frac{1}{3}}$

$$=\left[27\left(1+\frac{3}{27}\right)\right]^{1/3}=(27)^{1/3}\left(1+\frac{1}{9}\right)^{1/3}$$

$$=3\left(1+\frac{1}{9}\right)^{1/3}$$

$$=3\left[1+\frac{1}{3}.\frac{1}{9}+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{1}{9}\right)^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{1}{9}\right)^3+...\right]$$

$$=3\left[1+\frac{1}{3}.\frac{1}{9}-\frac{1}{9}\left(\frac{1}{9}\right)^2+\frac{5}{81}\left(\frac{1}{9}\right)^3+...\right]=3\left[1+\frac{1}{27}-\left(\frac{1}{27}\right)^2+...\right]$$

$$\approx 3[1+.03704-.001372]=3[1.035668]=3.107004$$

Thus $\sqrt{30}\approx 3.107$

Example 5: Find the coefficient of x^n in the expansion of $\frac{1-x}{(1+x)^2}$

Solution: $\frac{1-x}{(1+x)^2}=(1-x)(1+x)^{-2}$

$$=(-x+1)[1+(-2)x+\frac{(-2)(-3)}{2!}x^2+....+\frac{(-2)(-3)...(-2-r+1)}{r!}x^r+...]$$

$$=(-x+1)[1+(-1)2x+(-1)^2 3x^2+...+(-1)^r\times(r+1)x^r+...]$$

$$=(-x+1)[1+(-1)2x+(-1)^2 3x^2+....+(-1)^{n-1}nx^{n-1}+(-1)^n(n+1)x^n+...]$$

coefficient of $x^n=(-1)(-1)^{n-1}n+(-1)^n(n+1)$

$$=(-1)^n n+(-1)^n(n+1)$$

$$=(-1)^n[n+(n+1)]$$

$$=(\quad 1)+.(2\quad 1)$$

Example 6: If x is so small that its cube and higher power can be neglected, show that

$$\sqrt{\frac{1-x}{1+x}}\approx 1-x+\frac{1}{2}x^2$$

Solution: $\sqrt{\frac{1-x}{1+x}}=(1\quad x)+^{1/2}(1\quad x)^{-1/2}$

$$=\left[1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2+...\right]\left[1+\left(-\frac{1}{2}\right)x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2+...\right]$$

$$=\left[1-\frac{1}{2}x-\frac{1}{8}x^2+...\right]\left[1-\frac{1}{2}x+\frac{3}{8}x^2+...\right]$$

$$=\left[\left(1-\frac{1}{2}x+\frac{3}{8}x^2\right)+\left(-\frac{1}{2}x+\frac{1}{4}x^2\right)-\frac{1}{8}x^2+...\right]$$

$$=1-\left(\frac{1}{2}+\frac{1}{2}\right)x+\left(\frac{3}{8}+\frac{1}{4}-\frac{1}{8}\right)x^2+...$$

$$\approx 1-x+\frac{1}{2}x^2$$

Example 7: If *m* and *n* are nearly equal, show that

$$\left(\frac{5m-2n}{3n}\right)^{1/3}\approx\frac{m}{m+2n}+\frac{n+m}{3n}$$

Solution: Put $m = n + h$ (here h is so small that its square and higher powers can be neglected)

L.H.S. $= \left(\frac{5m-2n}{3n}\right)^{1/3} \quad \left(\frac{5(n+h)-2n}{3n}\right)^{1/3} \quad \left(\frac{3n+5h}{3n}\right)^{1/3}$

$= \left(1+\frac{5h}{3n}\right)^{1/3}$

$\approx 1+\frac{5h}{9h}$ (neglecting square and higher powers of h) (i)

R.H.S. $= \frac{m}{m+2n} + \frac{n+m}{3n}$

$= \frac{n+h}{3n+h} + \frac{2n+h}{3n}$

$= \frac{(n+h)}{3n}\left(\frac{1}{1+\frac{h}{3n}}\right) \quad \left(\frac{2}{3} \quad \frac{h}{3n}\right)$

$= (n+h)\frac{1}{3n}\left(1+\frac{h}{3n}\right)^{-1} + \left(\frac{2}{3}+\frac{h}{3n}\right)$

$= \left(\frac{1}{3}+\frac{h}{3n}\right)\left(1-\frac{h}{3n}+...\right)+\left(\frac{2}{3}+\frac{h}{3n}\right)$

$= \left[\frac{1}{3}+\left(-\frac{h}{9n}+\frac{h}{3n}\right)+...\right]+\frac{2}{3}+\frac{h}{3n}$

$\approx 1+\frac{5h}{9n}$ (neglecting square and higher powers of h) (ii)

From (i) and (ii), we have the result.

Example 8: Identify the series: $1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+...$ as a binomial expansion and find its sum

Solution: Let the given series be identical with.

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+... \qquad \text{(A)}$$

We know that (A) is expansion of $(1+x)^n$ for $|x|<1$ and n is not a positive integer. Now comparing the given series with (A) we get:

$$nx = \frac{1}{3} \qquad \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{3.6} \qquad \text{(ii)}$$

From (i), $x = \frac{1}{3n}$

Now substitution of $x = \frac{1}{3n}$ in (ii) gives

$$\frac{n(n-1)}{2!}.\left(\frac{1}{3n}\right)^2 = \frac{1}{6} \text{ or } \frac{n(n-1)}{2!}.\frac{1}{9n^2} \quad \frac{1}{6}$$

or $n-1 = 3n \Rightarrow n = -\frac{1}{2}$

Putting $n = -\frac{1}{2}$ in (iii), we get

$$x = \frac{1}{3\left(-\frac{1}{2}\right)} = -\frac{2}{3}$$

Thus the given series is the expansion of $\left[1+\left(-\frac{2}{3}\right)\right]^{-1/2}$ or $\left(1-\frac{2}{3}\right)^{-1/2}$

Hence the sum of the given series $= \left(1 - \frac{2}{3}\right)^{-1/2} = \left(\frac{1}{3}\right)^{-\frac{1}{2}} = (3)^{1/2}$

$= \sqrt{3}$

Example 9: For $y = \frac{1}{2}\left(\frac{4}{9}\right) + \frac{1.3}{2^2.2!}\left(\frac{4}{9}\right)^2 + \frac{1.3.5}{2^3.3!}\left(\frac{4}{9}\right)^3 + \ldots$

show that $5y^2 + 10y - 4 = 0$

Solution: $y = \frac{1}{2}\left(\frac{4}{9}\right) + \frac{1.3}{4.2!}\left(\frac{4}{9}\right)^2 + \frac{1.3.5}{8.3!}\left(\frac{4}{9}\right)^3 + \ldots$ (A)

Adding 1 to both sides of (A), we obtain

$$1 + y = 1 + \frac{1}{2}\left(\frac{4}{9}\right) + \frac{1.3}{4.2!}\left(\frac{4}{9}\right)^2 + \frac{1.3.5}{8.3!}\left(\frac{4}{9}\right)^3 + \ldots \quad \text{(B)}$$

Let the series on the right side of (B) be identical with

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

which is the expansion of $(1+x)^n$ for $|x| < 1$ and n is not a positive integer. On comparing terms of both the series, we get

$$nx = \frac{1}{2}.\left(\frac{4}{9}\right) \quad \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{4.2!}\left(\frac{4}{9}\right)^2 \quad \text{(ii)}$$

From (i), $x = \frac{2}{9n}$ (iii)

Substituting $x = \frac{2}{9n}$ in (ii), we get

$$\frac{n(n-1)}{2}.\left(\frac{2}{9n}\right)^2 = \frac{3}{8}.\frac{16}{81} \quad \text{or} \quad \frac{n(n-1)}{2}.\frac{4}{81n^2} = \frac{3}{8}.\frac{16}{81}$$

or $2(n - 1) = 6n$ or $n - 1 = 3n \Rightarrow n = -\frac{1}{2}$

Putting $n = -\frac{1}{2}$ in (iii), we get

$$x = \frac{2}{9\left(-\frac{1}{2}\right)} = -\frac{4}{9}$$

Thus $1 + y = \left(1 - \frac{4}{9}\right)^{-1/2} = \left(\frac{5}{9}\right)^{-1/2} = \left(\frac{9}{5}\right)^{1/2}$

$= \frac{3}{\sqrt{5}}$

or $\sqrt{5}(1+y) = 3$ (iv)

Squaring both the sides of (iv), we get

$5(1 + 2y + y^2) = 9$

or $5y^2 + 10y - 4 = 0$

Exercise 8.3

1. Expand the following upto 4 terms, taking the values of x such that the expansion in each case is valid.

i) $(1-x)^{1/2}$ ii) $(1+2x)^{-1}$ iii) $(1+x)^{-1/3}$ iv) $(4-3x)^{1/2}$

v) $(8-2x)^{-1}$ vi) $(2-3x)^{-2}$ vii) $\frac{(1-x)^{-1}}{(1+x)^2}$ viii) $\frac{\sqrt{1+2x}}{1-x}$

ix) $\frac{(4+2x)^{1/2}}{2-x}$ x) $(1+x-2x^2)^{\frac{1}{2}}$ xi) $(1-2x+3x^2)^{\frac{1}{2}}$

2. Using Binomial theorem find the value of the following to three places of decimals.

i) $\sqrt{99}$ ii) $(.98)^{\frac{1}{2}}$ iii) $(1.03)^{\frac{1}{3}}$ iv) $\sqrt[3]{65}$

v) $\sqrt[4]{17}$ vi) $\sqrt[5]{31}$ vii) $\frac{1}{\sqrt[3]{998}}$ viii) $\frac{1}{\sqrt[5]{252}}$

ix) $\frac{\sqrt{7}}{\sqrt{8}}$ x) $(.998)^{-\frac{1}{3}}$ xi) $\frac{1}{\sqrt[6]{486}}$ xii) $(1280)^{\frac{1}{4}}$

3. Find the coefficient of x^n in the expansion of

i) $\frac{1+x^2}{(1+x)^2}$ ii) $\frac{(1+x)^2}{(1-x)^2}$ iii) $\frac{(1+x)^3}{(1-x)^2}$

iv) $\frac{(1+x)^2}{(1-x)^3}$ v) $(1-x+x^2-x^3+...)^2$

4. If x is so small that its square and higher powers can be neglected, then show that

i) $\frac{1-x}{\sqrt{1+x}} \approx 1-\frac{3}{2}x$ ii) $\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1+\frac{3}{2}x$

iii) $\frac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x} \simeq \frac{1}{4}-\frac{17}{384}x$

iv) $\frac{\sqrt{4+x}}{(1-x)^3} \approx 2+\frac{25}{4}x$

v) $\frac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} \approx 4\left(1-\frac{5x}{6}\right)$

vi) $\frac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \frac{3}{2}-\frac{61}{48}x$

vii) $\frac{\sqrt{4-x}+(8-x)^{1/3}}{(8-x)^{1/3}} \approx 2-\frac{1}{12}x$

5. If x is so small that its cube and higher power can be neglected, then show that

i) $\sqrt{1-x-2x^2} \approx 1-\frac{1}{2}x-\frac{9}{8}x^2$ ii) $\sqrt{\frac{1+x}{1-x}} \approx 1+x+\frac{1}{2}x^2$

6. If x is very nearly equal 1, then prove that $px^p - qx^q \approx (p-q)x^{p+q}$

7. If $p-q$ is small when compared with p or q, show that

$$\frac{(2n+1)p+(2n-1))q}{(2n-1)p+(2n+1)q} \approx \left(\frac{p+q}{2q}\right)^{1/n}$$

8. Show that $\left[\frac{n}{2(n+N)}\right]^{1/2} \approx \frac{8n}{9n-N}-\frac{n+N}{4n}$ where n and N are nearly equal.

9. Identify the following series as binomial expansion and find the sum in each case.

i) $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2!4}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{3!8}\left(\frac{1}{4}\right)^3+...$

ii) $1-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1.3}{2.4}\left(\frac{1}{2}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)^3+...$

iii) $1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+...$

iv) $1-\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3+...$

10. Use binomial theorem to show that $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+...=\sqrt{2}$

11. If $y=\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3$ ..., then prove that $y^2+2y-2-0$

12. If $2y=\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\frac{1.3.5}{3!}.\frac{1}{2^6}$ ... then prove that $4y^2+4y-1=0$

13. If $y=\frac{2}{5}+\frac{1.3}{2!}.\left(\frac{2}{5}\right)^2+\frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3$ ... then prove that $y^2+2y-4=0$

Fundamentals of Trigonometry

9.1 Introduction

Trigonometry is an important branch of Mathematics. The word **Trigonometry** has been derived from three Greek words: ***Trei*** (three), ***Goni*** (angles) and ***Metron*** (measurement). Literally it means **measurement of triangle.**

For study of calculus it is essential to have a sound knowledge of trigonometry.

It is extensively used in Business, Engineering, Surveying, Navigation, Astronomy, Physical and Social Sciences.

9.2 Units of Measures of Angles

Concept of an Angle

Two rays with a common starting point form an angle. One of the rays of angle is called initial side and the other as terminal side. The angle is identified by showing the direction of rotation from the initial side to the terminal side.

An angle is said to be positive/negative if the rotation is anti-clockwise/clockwise. Angles are usually denoted by Greek letters such as α (alpha), β (beta), γ (gamma), θ (theta) etc.

In figure 9.1 $\angle AOB$ is positive and $\angle COD$ is negative.

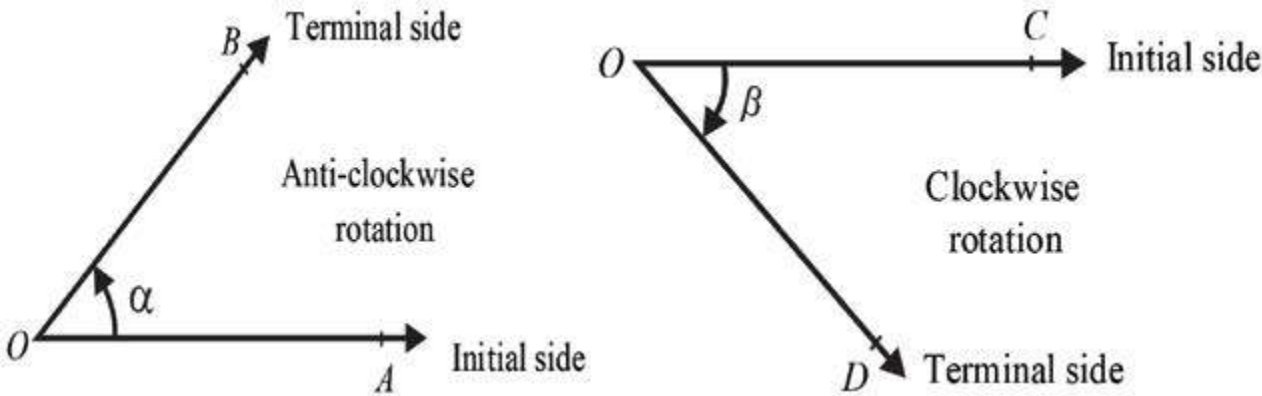



figure 9.1

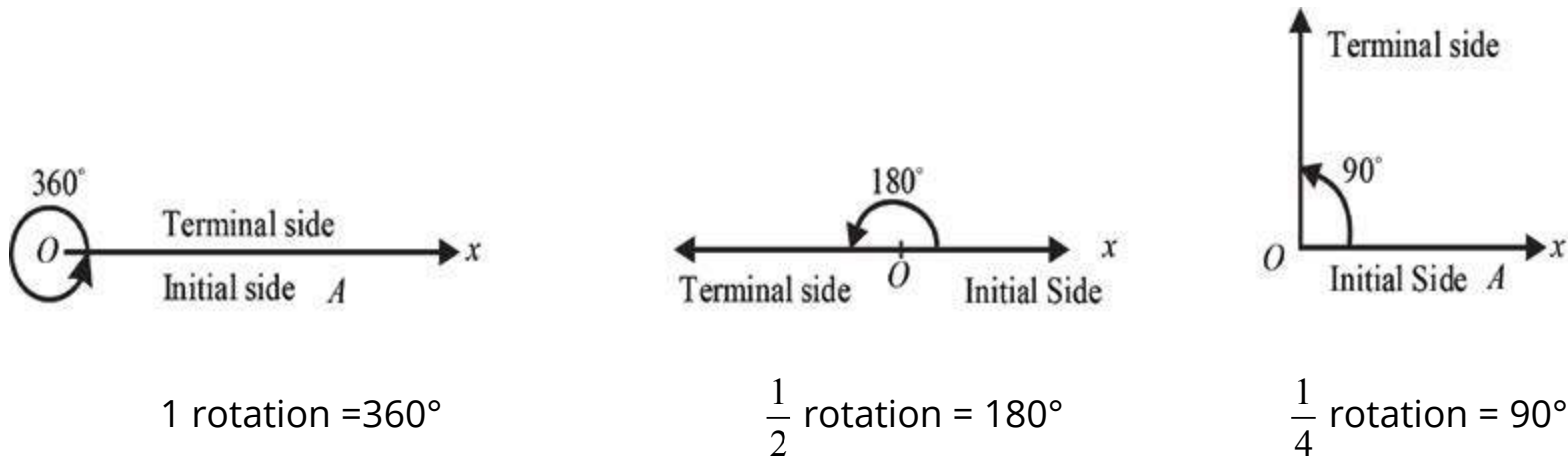
There are two commonly used measurements for angles: **Degrees and Radians.** which are explained as below:

9.2.1. Sexagesimal System: (Degree, Minute and Second).

If the initial ray $\overline{OA}$ rotates in anti-clockwise direction in such a way that it coincides with itself, the angle then formed is said to be of 360 degrees (360°).

One rotation (anti-clockwise) = 360°

$\frac{1}{2}$ rotation (anti-clockwise) = 180° is called a straight angle

$\frac{1}{4}$ rotation (anti-clockwise) = 90° is called a right angle.

1 degree (1°) is divided into 60 minutes ($60'$) and 1 minute ($1'$) is divided into 60 seconds ($60''$). As this system of measurement of angle owes its origin to the English and because 90, 60 are multiples of 6 and 10, so it is known as English system or Sexagesimal system.

Thus	1 rotation (anti-clockwise)	= 360°.
	One degree (1°)	= 60′
	One minute ($1'$)	= 60″

9.2.2. Conversion from $D^\circ M' S''$ to a decimal form and vice versa.

(i) $16^\circ 30' = 16.5^\circ$ (As $30' = \frac{1^\circ}{2}$ 0.5°)

(ii) $45.25^\circ = 45^\circ 15' \quad \left(0.25^\circ = \frac{25^\circ}{100} = \frac{1^\circ}{4} = \frac{60'}{4} = 15'\right)$

Example 1: Convert $18^\circ\ 6'\ 21''$ to decimal form.

Solution: $1' = \left(\frac{1}{60}\right)^\circ$ and $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{60\times 60}\right)^\circ$

$$\therefore\ 18^\circ\ 6'\ 21'' = \left[18 + 6\left(\frac{1}{60}\right) + 21\left(\frac{1}{60\times 60}\right)\right]^\circ$$

$= (18 + 0.1 + 0.005833)^\circ = 18.105833^\circ$

Example 2: Convert 21.256° to the $D^\circ\ M'\ S''$ form

Solution: $0.256^\circ = (0.256)(1^\circ)$

$= (0.256)(60') = 15.36'$

and $0.36' = (0.36)(1')$

$= (0.36)(60'') = 21.6''$

Therefore,

$21.256^\circ = 21^\circ + 0.256^\circ$

$= 21^\circ + 15.36'$

$= 21^\circ + 15' + 0.36'$

$= 21^\circ + 15' + 21.6''$

$= 21^\circ\ 15'\ 22''$ rounded off to nearest second

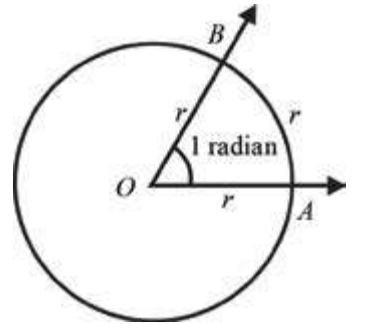
9.2.3. Circular System (Radians)

There is another system of angular measurement, called the **Circular System.** It is

most useful for the study of higher mathematics. Specially in Calculus, angles are measured in radians.

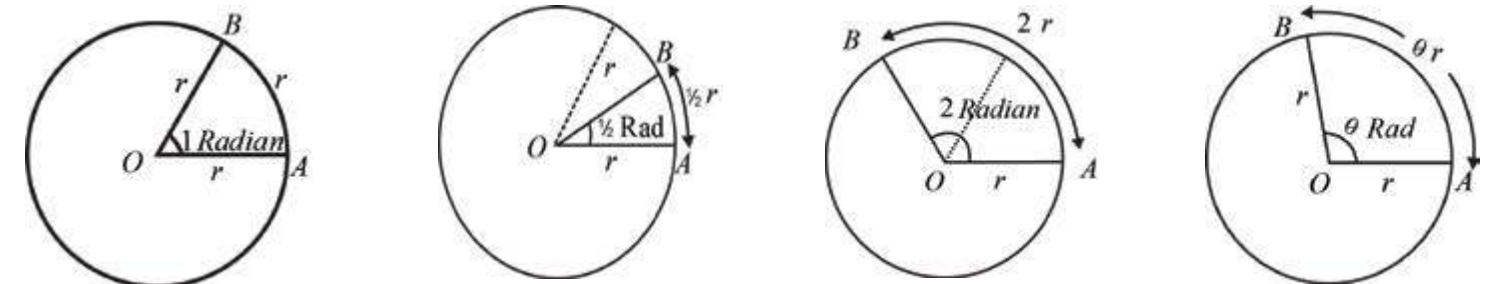
Definition: Radian is the measure of the angle subtended at the center of the circle by an arc, whose length is equal to the radius of the circle.

Consider a circle of radius r. Construct an angle $\angle AOB$ at the centre of the circle whose rays cut off an arc $\overset{\frown}{AB}$ on the circle whose length is equal to the radius r.

Thus $m\angle AOB = 1$ radian.

9.3 Relation between the length of an arc of a circle and the circular measure of its central angle.

Prove that $\theta = \frac{l}{r}$

where r is the radius of the circle l, is the length of the arc and θ is the circular measure of the central angle.

Proof:

By definition of radian;

An angle of 1 radian subtends an arc $\overset{\frown}{AB}$ on the circle of length $= 1.r$

An angle of $\frac{1}{2}$ radian subtends an arc $\overset{\frown}{AB}$ on the circle of length $= \frac{1}{2}.r$

An angle of 2 radians subtends an arc $\overset{\frown}{AB}$ on the circle of length $= 2.r$

$\therefore$ An angle of θ radian subtends an arc $\overset{\frown}{AB}$ on the circle of length $= \theta.r$

$$\Rightarrow \quad \overset{\frown}{AB} = \theta.r$$

$$\Rightarrow \quad l = \theta . r$$

$$\therefore \quad \theta = \frac{l}{r}$$

Alternate Proof

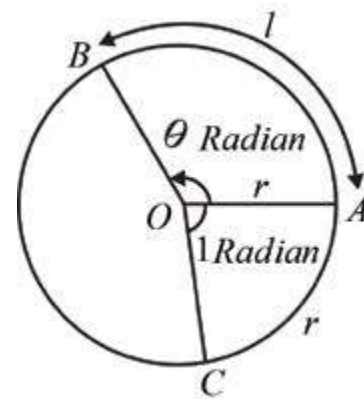
Let there be a circle with centre *O* and radius *r*. Suppose that length of arc $\overset{\frown}{AB} = l$ and the central angle $m\angle AOB = \theta$ radian. Take an arc $\overset{\frown}{AC}$ of length = r.

By definition $m\angle AOC = 1$ radian.

We know from elementary geometry that measures of central angles of the arcs of a circle are proportional to the lengths of their arcs.

$$\Rightarrow \quad \frac{m\angle AOB}{m\angle AOC} = \frac{m\overset{\frown}{AB}}{m\overset{\frown}{AC}}$$

$$\Rightarrow \quad \frac{\theta \text{ radian}}{1 \text{ radian}} = \frac{l}{r}$$

$$\Rightarrow \quad \theta = \frac{l}{r}$$

Thus the central angle θ (in radian) subtended by a circular arc of length l is given by $\theta = \frac{l}{r}$, where r is the radius of the circle.

Remember that r and l are measured in terms of the same unit and the radian measure is unit-less, i.e., it is a real number.

For example, if r = 3 cm and l = 6 cm

then $\quad \theta = \frac{l}{r} = \frac{6}{3} = 2$

9.3.1 Conversion of Radian into Degree and Vice Versa

We know that circumference of a circle of radius r is $2\pi r = (l)$, and angle formed by one complete revolution is θ radian, therefore,

$$\theta = \frac{l}{r}$$

$$\Rightarrow \quad \theta = \frac{2\pi r}{r}$$

$$\Rightarrow \quad \theta = 2\pi \text{ radian}$$

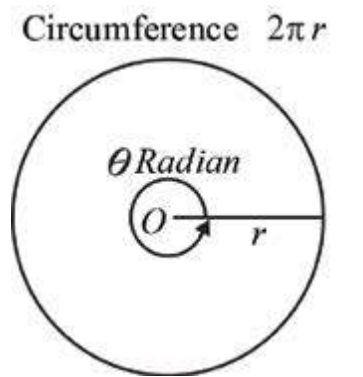
Thus we have the relationship

$$2\pi \text{ radian} = 360^\circ$$

$$\Rightarrow \quad \pi \text{ radian} = 180^\circ$$

$$\Rightarrow \quad 1 \text{ radian} = \frac{180^\circ}{\pi} \approx \frac{180^\circ}{3.1416} \approx 57.296^\circ$$

Further $\quad 1^\circ = \frac{\pi}{180}$ radian.

$$\approx \frac{3.1416}{180} \approx 0.0175 \text{ radain}$$

Example 3: Convert the following angles in degree:

(i) $\frac{2\pi}{3}$ radain (ii) 3 radians.

Solution: (i) $\frac{2\pi}{3}$ radains $= \frac{2}{3}(\pi \text{ radain}) \quad = \frac{2}{3}(180^\circ) \quad 120^\circ$

(ii) 3 radains = 3(1 radain) $\approx 3(57.296^\circ) \approx 171.888^\circ$

Example 4: Convert 54° 45 ʹ into radians.

Solution: $54^\circ 45' = \left(54\frac{45}{60}\right)^\circ = \left(54\frac{3}{4}\right)^\circ \quad \frac{219^\circ}{4}$

$$= \frac{219}{4}(1^\circ)$$

$$\approx \frac{219}{4}(0.0175) \text{ radinas}$$

$$\approx 0.958 \text{ radains.}$$

Most calculators automatically would convert degrees into radians and radians into degrees.

Example 5: An arc subtends an angle of 70° at the center of a circle and its length is 132 m.m. Find the radius of the circle.

Solution:

$$70^\circ = 70 \quad \frac{3.1416}{180} \text{radains} = \frac{70}{180}(3.1416)\text{radain} \quad \frac{11}{9}\text{radains.} \ (\pi \quad 3.1416)$$

$$\therefore \quad \theta = \frac{11}{9} \text{ radain} \quad \text{and} \quad l \quad 132\text{m.m.}$$

$$\because \quad \theta = \frac{l}{r} \quad \Rightarrow \quad r = \frac{l}{\theta} = 132 \times \frac{9}{11} = 108 \text{ m.m.}$$

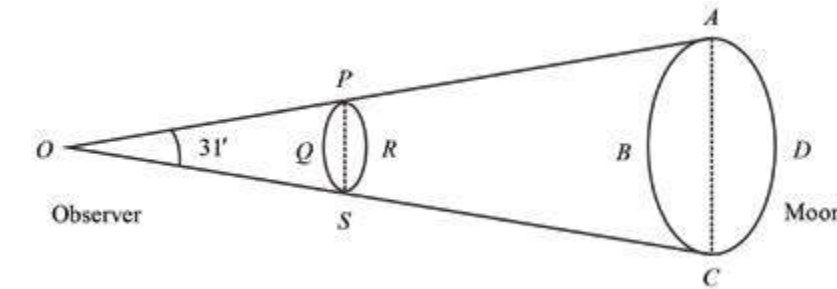
Example 6: Find the length of the equatorial arc subtending an angle of 1° at the centre of the earth, taking the radius of the earth as 6400 km.

Solution: $1^\circ = \approx \frac{\pi}{180} \text{ radains} \quad \frac{3.1416}{180} \text{ radain}$

$$\therefore \quad \theta \approx \frac{3.1416}{180} \quad \text{and} \quad r = 6400 \text{ km.}$$

Now $\theta = \frac{l}{r}$

$$\Rightarrow \quad l = r\theta \approx 6400 \times \frac{31416}{1800000} \approx 111.7 \text{ km}$$

Example 7: Find correct to the nearest centimeter, the distance at which a coin of diameter '1' cm should be held so as to conceal the full moon whose diameter subtends an angle of 31' at the eye of the observer on the earth.

Solution: Let *O* be the eye of the observer. *ABCD* be the moon and *PQSR* be the coin, so that *APO* and *CSO* are straight line segments.

We know that $m\overline{PS} = 1 \ cm, \quad m\angle AOC = 31'$

Now since $m\angle ACO(\cong m \quad POC)$ is very very small.

$\therefore \quad \overline{PS}$ can be taken as the arc .of the circle with centre *O* and radius *OP*.

Now $OP = r \ , \ l = 1 \text{ cm} \ , \ \theta = 31' \quad = \frac{31 \times \pi}{60 \times 180} \text{ radains}$

$$\because \quad \theta = \frac{l}{r}$$

$$\therefore \quad r = \frac{l}{\theta} = \frac{1 \times 60 \times 180}{31 \times \pi} \approx \frac{60 \times 180}{31 \times 3.1416} \approx 110.89 \ cm.$$

Hence the coin should be held at an approximate distance of 111 *cm*. from the observer's eye.

Note: If the value of π is not given, we shall take $\pi \approx 3.1416$.

Exercise 9.1

1. Express the following sexagesimal measures of angles in radians:

i) 30° ii) 45° iii) 60° iv) 75°
v) 90° vi) 105° vii) 120° viii) 135°
ix) 150° x) $10^\circ\ 15'$ xi) $35^\circ\ 20'$ xii) $75^\circ\ 6'\ 30''$
xiii) $120'\ 40''$ xiv) $154^\circ\ 20''$ xv) 0° xvi) $3''$

2. Convert the following radian measures of angles into the measures of sexagesimal system:

i) $\frac{\pi}{8}$ ii) $\frac{\pi}{6}$ iii) $\frac{\pi}{4}$ iv) $\frac{\pi}{3}$ v) $\frac{\pi}{2}$
vi) $\frac{2\pi}{3}$ vii) $\frac{3\pi}{4}$ viii) $\frac{5\pi}{6}$ ix) $\frac{7\pi}{12}$ x) $\frac{9\pi}{5}$
xi) $\frac{11\pi}{27}$ xii) $\frac{13\pi}{16}$ xiii) $\frac{17\pi}{24}$ xiv) $\frac{25\pi}{36}$ xv) $\frac{19\pi}{32}$

3. What is the circular measure of the angle between the hands of a watch at 4 *O*'clock?

4. Find θ, when:
i) l = 1.5 cm, r = 2.5cm
ii) l = 3.2m, r = 2m

5. Find l, when:
i) $\theta = \pi$ randains, r = 6cm
ii) $\theta = 65^\circ\ 20'$ r = 18mm

6. Find r, when:
i) l = 5 cm, $\theta = \frac{1}{2}$ radian
ii) l = 56 cm, $\theta = 45^\circ$

7. What is the length of the arc intercepted on a circle of radius 14 cms by the arms of a central angle of 45°?

8. Find the radius of the circle, in which the arms of a central angle of measure 1 radian cut off an arc of length 35 cm.

9. A railway train is running on a circular track of radius 500 meters at the rate of 30 km per hour. Through what angle will it turn in 10 sec.?

10. A horse is tethered to a peg by a rope of 9 meters length and it can move in a circle with the peg as centre. If the horse moves along the circumference of the circle, keeping the rope tight, how far will it have gone when the rope has turned through an angle of 70°?

11. The pendulum of a clock is 20 cm long and it swings through an angle of 20° each second. How far does the tip of the pendulum move in 1 second?

12. Assuming the average distance of the earth from the sun to be 148 x 10^6 km and the angle subtended by the sun at the eye of a person on the earth of measure 9.3 x 10^{-3} radian. Find the diameter of the sun.

13. A circular wire of radius 6 cm is cut straightened and then bent so as to lie along the circumference of a hoop of radius 24 cm. Find the measure of the angle which it subtends at the centre of the hoop.

14. Show that the area of a sector of a circular region of radius r is $\frac{1}{2}r^2\theta$, where θ is the circular measure of the central angle of the sector.

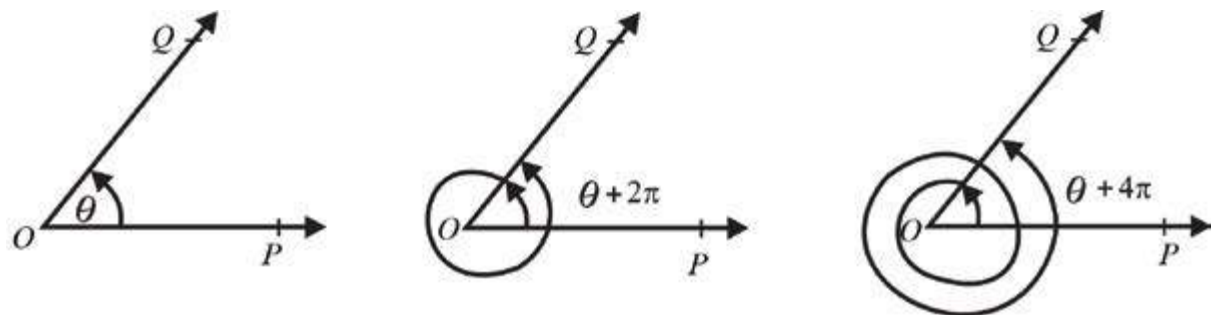
15. Two cities *A* and *B* lie on the equator such that their longitudes are 45°E and 25°W respectively. Find the distance between the two cities, taking radius of the earth as 6400 kms.

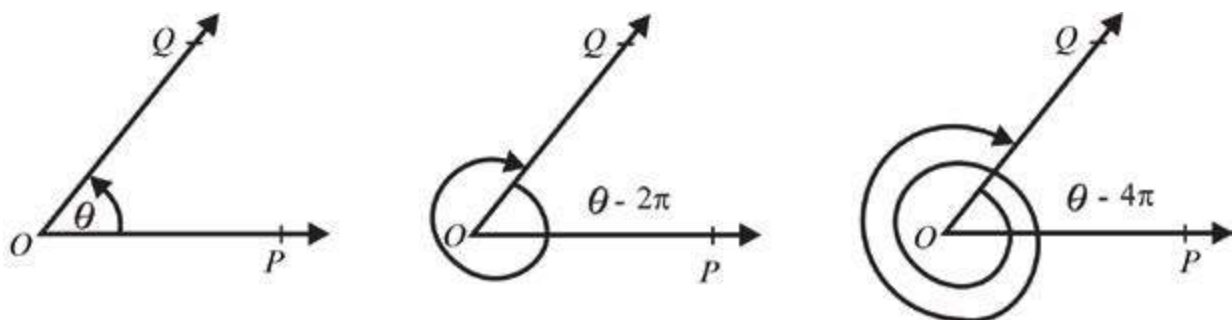
16. The moon subtends an angle of 0.5° at the eye of an observer on earth. The distance of the moon from the earth is 3.844 x 10^5 km approx. What is the length of the diameter of the moon?

17. The angle subtended by the earth at the eye of a spaceman, landed on the moon, is 1° 54'. The radius of the earth is 6400 km. Find the approximate distance between the moon and the earth.

9.4 General Angle (Coterminal Angles)

There can be many angles with the same initial and terminal sides. These are called coterminal angles. Consider an angle $\angle POQ$ with initial side $\overrightarrow{OP}$ and terminal side $\overrightarrow{OQ}$ with

vertex *O*. Let $m\angle POQ = \theta$ radian, where $0 \le \theta \le 2\pi$

Now, if the side $\overrightarrow{OQ}$ comes to its present position after one or more complete rotations in the anti-clockwise direction, then $m\angle POQ$ will be

i) $\theta + 2\pi$, after one revolution ii) $\theta + 4\pi$, after two revolutions,

However, if the rotations are made in the clock-wise direction as shown in the figure, $m\angle POQ$ will be:

i) $\theta - 2\pi$, after one revolution,

ii) $\theta - 4\pi$, after two revolution,

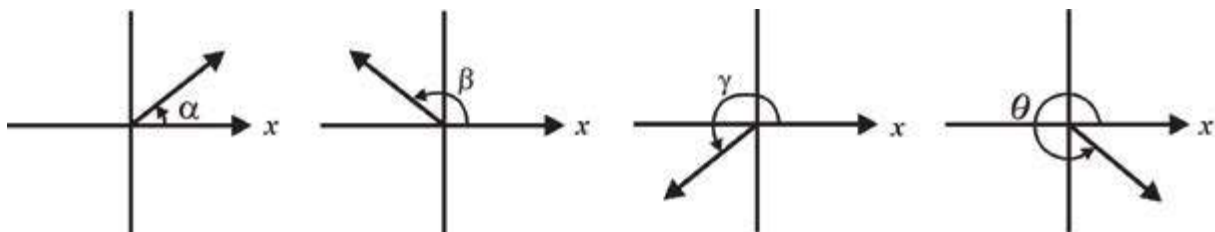
It means that $\overrightarrow{OQ}$ comes to its original position after every revolution of 2π radians in the postive or negative directions.

In general, if angle θ is in degrees, then $\theta + 360k$ *where* $k \in Z$, is an angle coterminal with θ. If angle θ is in radians, then $\theta + 2k\pi$ *where* $k \in Z$, is an angle coterminal with θ.

$\Rightarrow$ **General angle is** $\theta + 2k\pi, \ k \in Z$,

9.5 Angle In The Standard Position

An angle is said to be in **standard position** if its vertex lies at the origin of a rectangular coordinate system and its initial side along the positive x-axis.

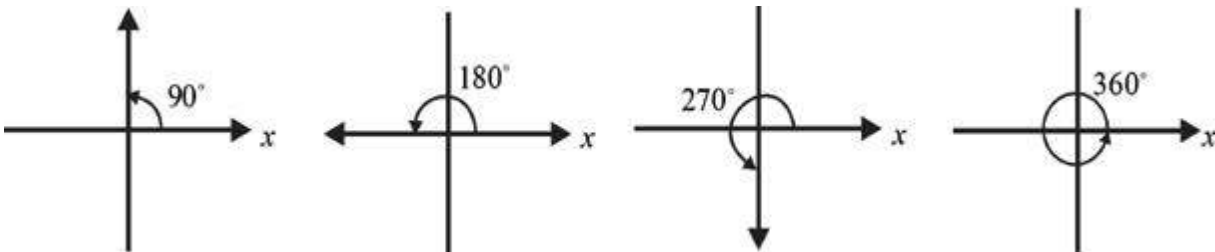
The following figures show four angles in standard position:

An angle in standard position is said to lie in a quadrant if its terminal side lies in that quadrant. In the above figure:

Angle α lies in I Quadrant as its terminal side lies is I Quadrant

Angle β lies in II Quadrant as its terminal side lies is II Quadrant

Angle γ lies in III Quadrant as its terminal side lies is III Quadrant

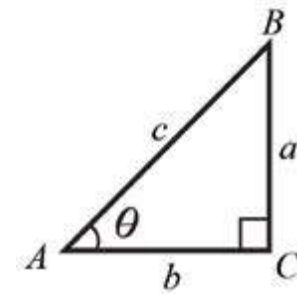
and Angle θ lies in IV Quadrant as its terminal side lies is IV Quadrant

If the terminal side of an angle falls on x-axis or y-axis, it is called a

quadrantal angle.

i.e., 90°, 180°, 270° and 360° are quadrantal angles.

9.6 Trigonometric Functions

Consider a right triangle *ABC* with $\angle C = 90^\circ$ and sides *a*, *b*, *c*, as shown in the figure. Let $m\angle A = \theta$ radian.

The side *AB* opposite to 90° is called the hypotenuse (hyp),
The side *BC* opposite to θ is called the opposite (opp) and
the side *AC* related to angle θ is called the adjacent (adj)

We can form six ratios as follows:

$$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{c}{a}, \frac{c}{b} \text{ and } \frac{b}{a}$$

In fact these ratios depend only on the size of the angle and not on the triangle formed. Therefore, these ratios are called **trigonometric functions** of angle θ and are defined as below:

$$\text{Sin}\,\theta \quad : \text{Sin}\,\theta = \frac{a}{c} = \frac{\text{opp}}{\text{hyp}}; \quad \text{Cosecant}\,\theta : \csc\theta = \frac{c}{a} = \frac{\text{hyp}}{\text{opp}};$$

$$\text{Cosine}\,\theta : \text{Cos}\,\theta = \frac{b}{c} = \frac{\text{adj}}{\text{hyp}}; \quad \text{Secant}\,\theta : \sec\theta = \frac{c}{b} = \frac{\text{hyp}}{\text{adj}};$$

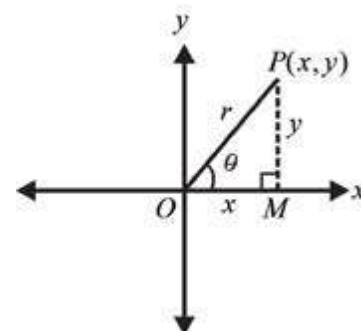
$$\text{Tangent}\,\theta : \tan\theta = \frac{a}{b} = \frac{\text{opp}}{\text{adj}}; \quad \text{Cotangent}\,\theta : \cot\theta = \frac{b}{a} = \frac{\text{adj}}{\text{opp}}.$$

We observe useful relationships between these **six trigonometric functions** as follows:

$$\csc\theta = \frac{1}{\sin\theta}; \quad \sec\theta = \frac{1}{\cos\theta}; \quad \tan\theta = \frac{\sin\theta}{\cos\theta};$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}; \quad \cot\theta = \frac{1}{\tan\theta};$$

9.7 Trigonometric Functions of any angle

Now we shall define the trigonometric functions of any angle.
Consider an angle $\angle XOP = \theta$ radian in standard position.
Let coordinates of P (other than origin) on the terminal side of

the angle be (x, y).

If $r = \sqrt{x^2 + y^2}$ denote the distance from O (0, 0) to P (x, y), then six trigonometric functions of θ are defined as the ratios

$$\sin\theta = \frac{y}{r}; \quad \csc\theta = \frac{r}{y} \ (y \neq 0); \quad \tan\theta = \frac{y}{x} \ (x \neq 0)$$

$$\cos\theta = \frac{x}{r}; \quad \sec\theta = \frac{r}{x} \ (x \neq 0); \quad \cot\theta = \frac{x}{y} \ (y \neq 0)$$

Note: These definitions are independent of the position of the point *P* on the terminal side i.e., θ is taken as any angle.

9.8 Fundamental Identities

For any real number θ, we shall derive the following three fundamental identities:

i) $\sin^2\theta + \cos^2\theta = 1$

ii) $1 + \tan^2\theta = \sec^2\theta$

iii) $1 + \cot^2\theta = \csc^2\theta.$

Proof:

(i) Refer to right triangle *ABC* in fig. (I) by Pythagoras theorem, we have, Dividing $a^2 + b^2 = c^2$ both sides by c^2, we get

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{a^2}$$

$$\Rightarrow \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$\Rightarrow (\sin\theta)^2 + (\cos\theta)^2 = 1$$

$$\therefore \boxed{\sin^2\theta + \cos^2\theta = 1} \qquad (1)$$

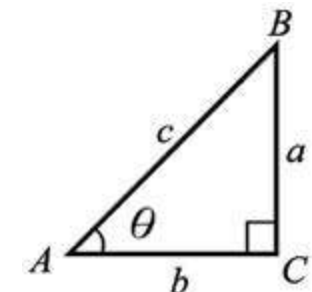



Fig.(1)

ii) Again as $a^2 + b^2 = c^2$
Dividing both sides by b^2, we get

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

$$\Rightarrow \left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

$$\Rightarrow (\tan\theta)^2 + 1 = (\sec\theta)^2$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta} \qquad (2)$$

(iii) Again as $a^2 + b^2 = c^2$
Dividing both sides by a^2, we get

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$

$$\Rightarrow 1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$$

$$\Rightarrow 1 + (\cot\theta)^2 = (\csc\theta)^2$$

$$\therefore \boxed{1 + \cot^2\theta = \csc^2\theta} \qquad (3)$$

Note: $(\sin\theta)^2 = \sin^2\theta$, $(\cos\theta)^2 = \cos^2\theta$ and $(\tan\theta)^2 = \tan^2\theta$ etc.

9.9 Signs of the Trigonometric functions

If θ is not a quadrantal angle, then it will lie in a particular quadrant. Because $r = \sqrt{x^2 + y^2}$ is always positive, it follows that the signs of the trigonometric functions can be found if the quadrant of 0 is known. For example,

(i) If θ lies in Quadrant I, then a point $P(x, y)$ on its terminal side has both x, y co-ordinates +ve

$\Rightarrow$ All trigonometric functions are +ve in Quadrant I.

(ii) If θ lies in Quadrant II, then a point $P(x, y)$ on its terminal side has negative x-coordinate. and positive y-coordinate.

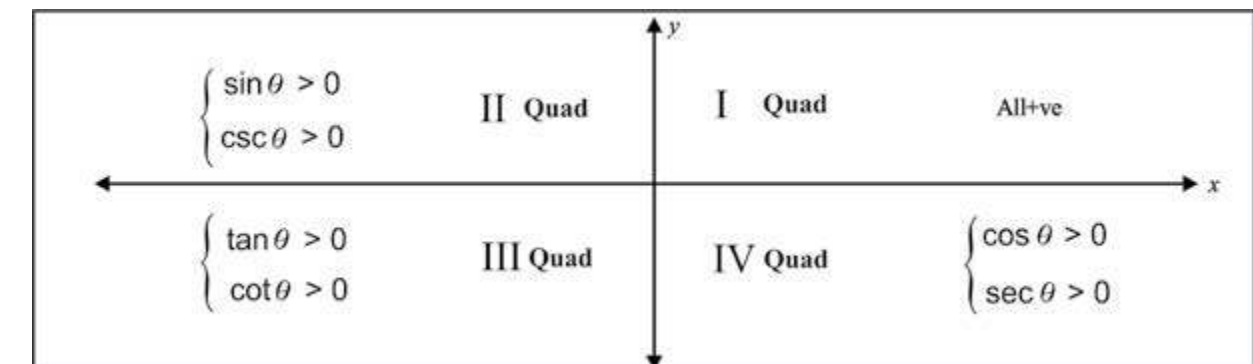
$$\therefore \sin\theta = \frac{y}{r} = +ve > 0, \cos\theta = \frac{x}{r} = -ve < 0, \tan\theta = \frac{y}{x} = -ve < 0$$

(iii) If θ lies in Quadrant III, then a point $P(x, y)$ on its terminal side has negative x-coordinate. and negative y-coordinate.

$$\therefore \sin\theta = \frac{y}{r} = -ve < 0, \cos\theta = \frac{x}{r} = -ve < 0, \tan\theta = \frac{y}{x} = +ve > 0$$

(iv) If θ lies in Quadrant IV, then a point $P(x, y)$ on its terminal side has positive x-coordinate. and negative y-coordinate.

$$\therefore \sin\theta = \frac{y}{r} = -ve < 0, \cos\theta = \frac{x}{r} = +ve > 0, \tan\theta = -ve < 0$$

These results are summarized in the following figure. Trigonometric functions mentioned are positive in these quardrants.

It is clear from the above figure that

$\sin(-\theta) = -\sin\theta$; $\qquad \csc(-\theta) = -\csc\theta$

$\cos(-\theta) = \cos\theta$; $\qquad \sec(-\theta) = \sec\theta$

$\tan(-\theta) = -\tan\theta$; $\qquad \cot(-\theta) = -\cot\theta$

Example 1: If $\tan\theta = \frac{8}{15}$ and the terminal arm of the angle is in the III quadrant, find the values of the other trigonometric functions of θ.

Solution:

$$\tan\theta = \frac{8}{15} \qquad \therefore \cot\theta = \frac{1}{\tan\theta} = \frac{15}{8}$$

$$\sec^2\theta = 1+\tan^2\theta = 1+\left(\frac{8}{15}\right)^2 = 1+\frac{64}{225} = \frac{289}{225}$$

$$\therefore \sec\theta = \pm\sqrt{\frac{289}{225}} = \pm\frac{17}{15}$$

$\because$ The terminal arm of the angle is in the III quadrant where sec θ is negative

$$\therefore \sec\theta = -\frac{17}{15}$$

Now $$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{-\frac{17}{15}} = -\frac{15}{17}$$

$$\sin\theta = \tan\theta . \cos\theta = \frac{8}{15}\left(-\frac{15}{17}\right)$$

$$\therefore \sin\theta = -\frac{8}{17}$$

and $$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{8}{17}} = -\frac{17}{8}$$

Example 2: Find the value of other five trigonometric functions of θ, if $\cos\theta = \frac{12}{13}$ and the terminal side of the angle is not in the I quadrant.

Solution: The terminal side of the angle is not in the I quadrant but cos θ is positive,

$\therefore$ The terminal side of the angle is in the IV quadrant

Now $$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$

$$\sin^2\theta = 1-\cos^2\theta = 1-\left(\frac{12}{13}\right)^2 = 1-\frac{144}{169} = \frac{25}{169}$$

$$\therefore \sin\theta = \pm\frac{5}{13}$$

As the terminal side of the angle is in the IV quadrant where sin θ is negative.

$$\therefore \sin\theta = -\frac{5}{13}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5}$$

Exercise 9.2

1. Find the signs of the following:

i) $\sin 160^\circ$ ii) $\cos 190^\circ$ iii) $\tan 115^\circ$

iv) $\sec 245^\circ$ v) $\cot 80^\circ$ vi) $\text{cosec}\, 297^\circ$

2. Fill in the blanks:

i) $\sin(-310^\circ) = ... \sin 310^\circ$ ii) $\cos(-75^\circ) = ... \cos 75^\circ$

iii) $\tan(-182^\circ) = ... \tan 182^\circ$ iv) $\cot(-173^\circ) = ... \cot 137^\circ$

v) $\sec(-216^\circ) = ... \sec 216^\circ$ vi) $\text{cosec}(-15^\circ) = ... \text{cosec}\, 15^\circ$

3. In which quadrant are the terminal arms of the angle lie when

i) $\sin\theta < 0$ and $\cos\theta > 0$, ii) $\cot\theta > 0$ and $\text{cosec}\,\theta > 0$,

iii) $\tan\theta < 0$ and $\cos\theta > 0$, iv) $\sec\theta < 0$ and $\sin\theta < 0$,

v) $\cot\theta > 0$ and $\sin\theta < 0$, vi) $\cos\theta < 0$ and $\tan\theta < 0$?

4. Find the values of the remaining trigonometric functions:

i) $\sin\theta = \frac{12}{13}$ and the terminal arm of the angle is in quad. I.

ii) $\cos\theta = \frac{9}{41}$ and the terminal arm of the angle is in quad. IV.

iii) $\cos\theta = -\frac{\sqrt{3}}{2}$ and the terminal arm of the angle is in quad. III.

iv) $\tan \quad = - -$ and the terminal arm of the angle is in quad. II.

v) $\sin\theta = -\frac{1}{\sqrt{2}}$ and the terminal arm of the angle is not in quad. III.

5. Find $\cot\theta = \frac{15}{8}$ and the terminal arm of the angle is not is quad. I, find the values of $\cos\theta$ and cosec θ.

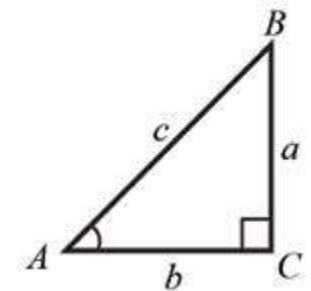
6. If $\text{cosec}\ \theta = \frac{m^2+1}{2m}$ and $m\rangle 0\left(0 < \theta < \frac{\pi}{2}\right)$, find the values of the remaining trigonometric

ratios.

7. If $\tan\theta = \frac{1}{\sqrt{7}}$ and the terminal arm of the angle is not in the III quad., find the values of $\frac{\csc^2\theta - \sec^2\theta}{\csc^2\theta + \sec^2\theta}$.

8. If $\cot\theta = \frac{5}{2}$ and the terminal arm of the angle is in the I quad., find the value of $\frac{3\sin\theta + 4\cos\theta}{\cos\theta - \sin\theta}$.

9.10 The values of Trigonometric Functions of acute angles 45°, 30° and 60°

Consider a right triangle *ABC* with $m\angle C = 90^\circ$ and sides *a*, *b*, *c* as shown in the figure on right hand side.

(a) **Case 1** when $m\angle A = 45^\circ = \frac{\pi}{4}$ randian

then $m\angle B = 45^\circ$

$\Rightarrow \Delta ABC$ is right isosceles.

As values of trigonometric functions depend only on the angle and not on the size of the triangle, we can take *a* = *b* = 1

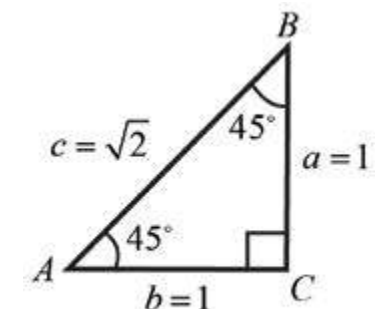
By Pythagoras theorem,

$$c^2 = a^2 + b^2$$

$$\Rightarrow \quad c^2 = 1 + 1 = 2$$

$$\Rightarrow \quad c = \sqrt{2}$$

Fig (1)

$\therefore$ Using triangle of fig. 1, with *a*=*b*= 1 and *c* = $\sqrt{2}$

$$\sin 45^\circ = \frac{a}{c} = \frac{1}{\sqrt{2}}; \qquad \csc 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2};$$

$$\cos 45^\circ = \frac{b}{c} = \frac{1}{\sqrt{2}}; \qquad \sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2};$$

$$\tan 45^\circ = \frac{a}{b} = 1; \qquad \cot 45^\circ = \frac{1}{\tan 45^\circ} = 1.$$

(b) **Case 2:** when $m\angle A = 30^\circ = \frac{\pi}{6}$ randian

then $m\angle B = 60^\circ$

By elementary geometry, in a right triangle the measure of the side opposite to 30° is half of the hypotenuse.

Let c = 2 then a = 1

∴ By Pythagoras theorem , $a^2+b^2 = c^4$

$$\Rightarrow b^2 = c^2 - a^2$$
$$= 4 - 1$$
$$= 3$$
$$\Rightarrow b = \sqrt{3}$$

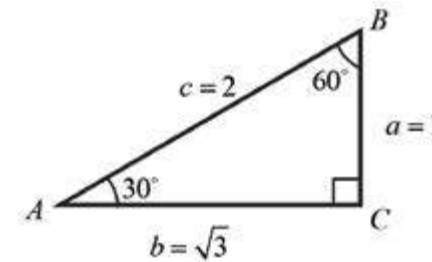



Fig (2)

∴ Using triangle of fig.2, with a = 1, $b = \sqrt{3}$ and c = 2

$$\sin 30^\circ = \frac{a}{c} = \frac{1}{2}; \qquad \csc 30^\circ = \frac{1}{\sin 30^\circ} = 2;$$

$$\cos 30^\circ = \frac{b}{c} = \frac{\sqrt{3}}{2}; \qquad \sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}};$$

$$\tan 30^\circ = \frac{a}{b} = \frac{1}{\sqrt{3}}; \qquad \cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}.$$

(c) **Case 3:** when $m\angle A = 60^\circ = \frac{\pi}{3}$ radian, then $m\angle B = 30^\circ$

By elementary geometry, in a right triangle the measure of the side opposite to 30° is

half the hypotenuse.

Let c = 2 then b = 1

∴ By Pythagoras theorem ,

$$\therefore \quad a^2 + b^2 = c^2$$
$$\Rightarrow a^2 = c^2 - b^2$$
$$= 4 - 1 = 3$$
$$\Rightarrow a = \sqrt{3}$$

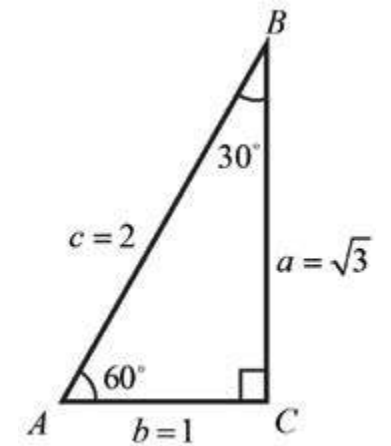



Fig (3)

∴ Using triangle of fig.3, with $a = \sqrt{3}$, , b = 1 and c = 2

$$\sin 60^\circ = \frac{a}{c} = \frac{\sqrt{3}}{2}; \qquad \csc 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}};$$

$$\cos 60^\circ = \frac{b}{c} = \frac{1}{2}; \qquad \sec 60^\circ = \frac{1}{\cos 60^\circ} = 2;$$

$$\tan 60^\circ = \frac{a}{b} = \sqrt{3}; \qquad \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}.$$

Example 3: Find the values of all the trigonometric functions of

(i) 420° (ii) $\frac{-7\pi}{4}$ (iii) $\frac{19\pi}{3}$

Solution: We know that $\theta + 2k\pi = \theta$, where $k \in Z$

(i) 420° = 60°+ 1(360°) (k= 1)

= 60°

$$\therefore \sin 420^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}; \qquad \csc 420^\circ = \frac{2}{\sqrt{3}}$$

$$\cos 420^\circ = \cos 60^\circ = \frac{1}{2}; \qquad \sec 420^\circ = 2$$

$$\tan 420^\circ = \tan 60^\circ = \sqrt{3}; \qquad \cot 420^\circ = \frac{1}{\sqrt{3}}$$

(ii) $\dfrac{-7\pi}{4} = \dfrac{\pi}{4} + (-1)2\pi \qquad (k = -1)$

$= \dfrac{\pi}{4}$

$\therefore \sin\left(\dfrac{-7\pi}{4}\right) = \sin\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}; \qquad \csc\left(\dfrac{-7\pi}{4}\right) = \csc\left(\dfrac{\pi}{4}\right) = \sqrt{2};$

$\cos\left(\dfrac{-7\pi}{4}\right) = \cos\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}; \qquad \sec\left(\dfrac{-7\pi}{4}\right) = \sec\left(\dfrac{\pi}{4}\right) = \sqrt{2};$

$\tan\left(\dfrac{-7\pi}{4}\right) = \tan\left(\dfrac{\pi}{4}\right) = 1; \qquad \cot\left(\dfrac{-7\pi}{4}\right) = \cot\left(\dfrac{\pi}{4}\right) = 1;$

(iii) $\dfrac{19\pi}{3} = \dfrac{\pi}{3} + 3(2\pi) \qquad (k = 3)$

$= \dfrac{\pi}{3}$

$\therefore \sin\left(\dfrac{19\pi}{3}\right) = \sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}; \qquad \csc\left(\dfrac{19\pi}{3}\right) = \csc\left(\dfrac{\pi}{3}\right) = \dfrac{2}{\sqrt{3}};$

$\cos\left(\dfrac{19\pi}{3}\right) = \cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}; \qquad \sec\left(\dfrac{19\pi}{3}\right) = \sec\left(\dfrac{\pi}{3}\right) = 2;$

$\tan\left(\dfrac{19\pi}{3}\right) = \tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}; \qquad \cot\left(\dfrac{19\pi}{3}\right) = \cot\left(\dfrac{\pi}{3}\right) = \dfrac{1}{\sqrt{3}}.$

9.11 The values of the Trigonometric Functions of angles 0°, 90°, 180°, 270°, 360°.

When terminal line lies on the x– *axis* or the y– *axis*, the angle θ is called a quadrantal angle.

Now we shall find the values of trigonometric functions of quadrantal angles 0°, 90°, 180°, 270°, 360° and so on.

(a) **When θ = 0°**

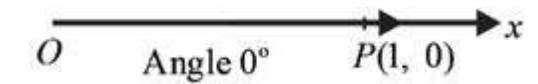
The point (1,0) lies on the terminal side of angle 0°

$\Rightarrow \quad x = 1$ and $y = 0$

so $\quad r = \sqrt{x^2 + y^2} = 1$

$\therefore \sin 0^\circ = \dfrac{y}{r} = \dfrac{0}{1} = 0 \qquad \csc 0^\circ = \dfrac{1}{\sin 0^\circ} = \dfrac{1}{0}$ (undefined)

$\cos 0^\circ = \dfrac{x}{r} = \dfrac{1}{1} = 1 \qquad \sec 0^\circ = \dfrac{1}{\cos 0^\circ} = 1$

$\tan 0^\circ = \dfrac{y}{x} = \dfrac{0}{1} = 0 \qquad \cot 0^\circ = \dfrac{1}{\tan 0^\circ} = \dfrac{1}{0}$ (undefined)

(b) **When $\theta = 90^\circ$**

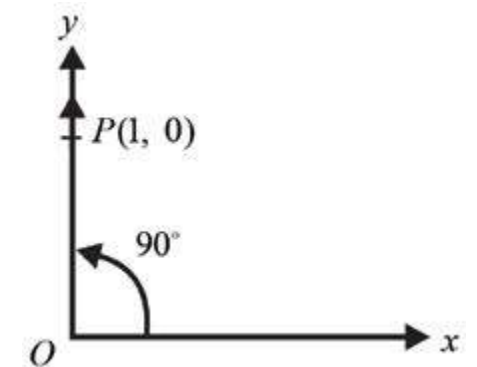
The point (0, 1) lies on the terminal side of angle 90°.

$\Rightarrow \quad x = 0$ and $y = 1$

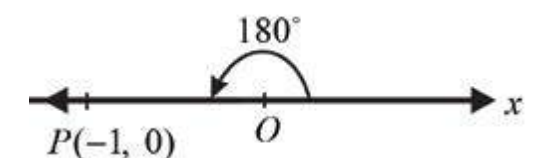
so $\quad r = \sqrt{x^2 + y^2} = 1$

$\therefore \sin 90^\circ = \dfrac{y}{r} = \dfrac{1}{1} = 1; \qquad \csc 90^\circ = \dfrac{1}{\sin 90^\circ} = 1;$

$\cos 90^\circ = \dfrac{x}{r} = \dfrac{0}{1} = 0; \qquad \sec 90^\circ = \dfrac{1}{\cos 90^\circ} = \dfrac{1}{0}$ (undefined);

$\tan 90^\circ = \dfrac{y}{x} = \dfrac{1}{0}$ (undefined); $\qquad \cot 90^\circ = \dfrac{x}{y} = \dfrac{0}{1} = 0.$

(c) **When $\theta = 180^\circ$**

The point (-1, 0) lies on the terminal side of angle 180°.

$\Rightarrow \quad x = -1$ and $y = 0$

so $r = \sqrt{x^2 + y^2} = 1$

$\therefore \sin 180^\circ = \frac{y}{r} = \frac{0}{1} \quad 0;$ $\qquad \csc 180^\circ \quad \frac{r}{y} \quad \frac{1}{0}$ (undefined);

$\cos 180^\circ = \frac{x}{r} \quad \frac{-1}{1} \quad 1;$ $\qquad \sec 180^\circ \quad \frac{r}{x} \quad \frac{1}{-1} \quad 1;$

$\tan 180^\circ = \frac{y}{x} = \frac{0}{-1} = 0;$ $\qquad \cot 180^\circ \quad \frac{x}{y} \quad \frac{1}{0}$ (undefined).

(d) **When $\theta = 270^\circ$**

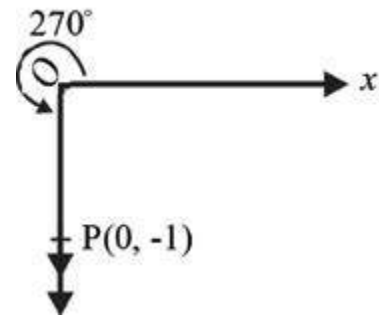
The point (0, -1) lies on the terminal side of angle 270°.

$\Rightarrow$ $x = 0$ and $y = -1$

so $r = \sqrt{x^2 + y^2} = 1$

$\therefore \sin 270^\circ = \frac{y}{r} \quad \frac{-1}{1} \quad 1;$ $\qquad \csc 270^\circ = \frac{r}{y} \quad \frac{1}{-1} \quad 1;$

$\cos 270^\circ = \frac{x}{r} = \frac{0}{1} = 0;$ $\qquad \sec 270^\circ \quad \frac{r}{x} \quad \frac{1}{0}$ (undefined);

$\tan 270^\circ = \frac{y}{x} = \frac{-1}{0}$ (undefined); $\qquad \cot 270^\circ = \frac{x}{y} = \frac{0}{-1} \quad 0.$

Example 4: Find the values of all trigonometric functions of

(i) 360° (ii) $\frac{-\pi}{2}$ (iii) 5π

Solution: We know that $\theta + 2k\pi = \theta$, where $k \in Z$

(i) Now 360° = 0° + 1(360°), (k = 1)
= 0°

$\sin 360^\circ = \sin 0^\circ = 0;$ $\qquad \csc 360^\circ$ is undefined;

$\cos 360^\circ = \cos 0^\circ = 1;$ $\qquad = \sec 360^\circ \quad \frac{1}{\cos 0^\circ} \quad 1$

$\tan 360^\circ = \tan 0^\circ = 0;$ $\qquad \cot 360^\circ$ is undefined.

(ii) We know that $\theta + 2k\pi = \theta$, where $k \in Z$

Now $-\frac{\pi}{2} = \frac{3\pi}{2} + (-1)2\pi \qquad (k = -1)$

$= \frac{3\pi}{2}$

$\therefore \sin\left(-\frac{\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1;$ $\qquad \csc\left(\frac{\pi}{2}\right) - 1; =$

$\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0; -$ $\qquad \sec\left(\frac{\pi}{2}\right)$ is undefined;

$\tan\left(-\frac{\pi}{2}\right) = \tan\left(\frac{3\pi}{2}\right)$ is undefined; $\qquad \cot\left(\frac{\pi}{2}\right) \quad 0$

(iii) Now $5\pi = \pi \quad 2(2\pi) \qquad (k \quad 2)$

$= \pi$

$\therefore \sin 5\pi = \sin \pi = 0;$ $\qquad \csc 5\pi$ is undefined;

$\cos 5\pi = \cos \pi \quad 1;$ $\qquad \sec 5\pi \quad 1;$

$\tan 5\pi = \tan \pi = 0;$ $\qquad \cot 5\pi$ is undefined;

Exercise 9.3

1. Verify the following:

(i) sin60° cos 30° – cos 60° sin 30° = sin 30°

(ii) $\sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{3} + \tan^2 \frac{\pi}{4} = 2$

(iii) $2\sin 45^\circ + \frac{1}{2}\text{cosec } 45^\circ = \frac{3}{\sqrt{2}}$

(iv) $\sin^2\frac{\pi}{6} : \sin^2\frac{\pi}{4} : \sin^2\frac{\pi}{3} : \sin^2\frac{\pi}{2} = 1:2:3:4.$

2. Evaluate the following:

i) $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$ ii) $\dfrac{1 - \tan^2\frac{\pi}{3}}{1 + \tan^2\frac{\pi}{3}}$

3. Verify the following when θ = 30°, 45°

i) $\sin 2\theta = 2\sin\theta\cos\theta$ ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

iii) $\cos 2\theta = 2\cos^2\theta - 1$ iv) $\cos 2\theta = 1 - 2\sin^2\theta$

v) $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

4. Find x, if $\tan^2 45° - \cos^2 60° = x \sin 45° \cos 45° \tan 60°$.

5. Find the values of the trigonometric functions of the following quadrantal angles:

i) $-\pi$ ii) -3π iii) $\frac{5}{2}\pi$

iv) $-\frac{9}{2}\pi$ v) -15π vi) 1530°

vii) – 2430° viii) $\frac{235}{2}\pi$ ix) $\frac{407}{2}\pi$

6. Find the values of the trigonometric functions of the following angles:

i) 390° ii) – 330° iii) 765°

iv) –675° v) $\frac{-17}{3}\pi$ vi) $\frac{13}{3}\pi$

vii) $\frac{25}{6}\pi$ viii) $\frac{-71}{6}\pi$ ix) –1035°

9.12 Domains of Trigonometric functions and of Fundamental Identities

We list the trigonometric functions and fundamental identities, learnt so far mentioning their domains as follows:

(i) $\sin\theta$, *for all* $\theta \in R$

(ii) $\cos\theta$, *for all* $\theta \in R$

(iii) $\csc\theta = \dfrac{1}{\sin\theta}$, *for all* $\theta \in R$ but $\theta \neq n\pi$, $n \in Z$

(iv) $\sec\theta = \dfrac{1}{\cos\theta}$, *for all* $\theta \in R$ but $\theta \neq \left(\dfrac{2n+1}{2}\right)\pi$, $n \in Z$

(v) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, *for all* $\theta \in R$ but $\theta \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

(vi) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, *for all* $\theta \in R$ but $\theta \neq n\pi$, $n \in Z$

(vii) $\sin^2\theta + \cos^2\theta = 1$, *for all* $\theta \in R$

(viii) $1 + \tan^2\theta = \sec^2\theta$, *for all* $\theta \in R$ but $\theta \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

(ix) $1 + \cot^2\theta = \csc^2\theta$, *for all* $\theta \in R$ but $\theta \neq n\pi$, $n \in Z$

Now we shall prove quite a few more identities with the help of the above mentioned identities.

Example 1: Prove that $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$, for all $\theta \in R$

Solution: L.H.S $= \cos^4\theta - \sin^4\theta$

$= \left(\cos^2\theta\right)^2 - \left(\sin^2\theta\right)^2$

$= \left(\cos^2\theta \quad \sin^2\theta\right)\left(\cos^2\theta \quad \sin^2\theta\right)$

$= (1)\left(\cos^2\theta - \sin^2\theta\right)$ $(\because \sin^2\theta \quad \cos^2\theta \quad 1)$

$= \cos^2\theta - \sin^2\theta = \text{R.H.S.}$

Hence $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

Example 2: Prove that $\sec^2 A + \text{cosec}^2 A = \sec^2 A \ \text{cosec}^2 A$ $\left(\textbf{Where } A \neq \frac{n\pi}{2}, n \in Z\right)$

Solution: L.H.S $= \sec^2 A + \text{cosec}^2 A$

$= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{\sin^2 A + \cos^2 A}{\cos^2 A \sin^2 A}$

$= \frac{1}{\cos^2 A \sin^2 A} =$ $[\because \sin^2 A \quad \cos^2 A \quad 1]$

$= \frac{1}{\cos^2 A} \cdot \frac{1}{\sin^2 A}$

$== \sec^2 A . \text{cosec}^2 A$ R.H.S

Hence $\sec^2 A + \text{cosec}^2 A = \sec^2 A . \text{cosec}^2 A.$

Example 3: Prove that $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$, where θ is not an odd multiple of $\frac{\pi}{2}$.

Solution:

L.H.S. $= \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$

$= \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} \sqrt{\frac{1-\sin\theta}{1-\sin\theta}}$ (rationalizing.)

$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$

$== \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}} \quad \frac{1-\sin\theta}{\cos\theta}$

$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{R.H.S}$

Hence $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta.$

Example 4: Show that $\cot^4\theta + \cot^2\theta = \text{cosec}^4\theta - \text{cosec}^2\theta$, where θ is not an integral multiple of $\frac{\pi}{2}$.

Solution:

L.H.S. $= \cot^4\theta + \cot^2\theta$

$=+ \cot^2\theta(\cot^2\theta \quad 1)$

$= (\text{cosec}^2\theta - 1)\text{cosec}^2\theta$

$= \text{cosec}^4\theta - \text{cosec}^2\theta$

$= \text{R.H.S.}$

Hence $\cot^4\theta + \cot^2\theta = \text{cosec}^4\theta - \text{cosec}^2\theta.$

Exercise 9.4

Prove the following identities, state the domain of θ in each case:

1. $\tan\theta + \cot\theta = \text{cosec}\,\theta \sec\theta$ **2.** $\sec\theta \ \text{cosec}\,\theta \sin\theta \cos\theta = 1$

3. $\cos\theta + \tan\theta\sin\theta = \sec\theta$

4. $\operatorname{cosec}\theta + \tan\theta\sec\theta = \operatorname{cosec}\theta\sec^2\theta$

5. $\sec^2\theta - \operatorname{cosec}^2\theta = \tan^2\theta - \cot^2\theta$

6. $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta$

7. $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

8. $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

9. $\cos^2\theta - \sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

10. $\dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \dfrac{\cot\theta - 1}{\cot\theta + 1}$

11. $\dfrac{\sin\theta}{1+\cos\theta} + \cot\theta = \operatorname{cosec}\theta$

12. $\dfrac{\cot^2\theta - 1}{1+\cot^2\theta} = 2\cos^2\theta - 1$

13. $\dfrac{1+\cos\theta}{1-\cos\theta} = (\operatorname{cosec}\theta + \cot\theta)^2$

14. $(\sec\theta - \tan\theta)^2 = \dfrac{1-\sin\theta}{1+\sin\theta}$

15. $\dfrac{2\tan\theta}{1+\tan^2\theta} = 2\sin\theta\cos\theta$

16. $\dfrac{1-\text{sion}\,\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

17. $(\tan\theta + \cot\theta)^2 = \sec^2\theta\operatorname{cosec}^2\theta$

18. $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$

19. $\dfrac{1}{\operatorname{cosec}\theta - \cot\theta} - \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{1}{\operatorname{cosec}\theta + \cot\theta}$

20. $\sin^3\theta - \cos^3\theta = (\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta)$

21. $\sin^6\theta - \cos^6\theta = (\sin^2\theta - \cos^2\theta)(1 - \sin^2\theta\cos^2\theta)$

22. $\sin^6\theta + \cos^6\theta = 1 \quad 3\sin^2\theta\cos^2\theta$

23. $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2\sec^2\theta$

24. $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} + \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \dfrac{2}{1-2\sin^2\theta}$

CHAPTER

10

Trigonometric Identities

10.1 Introduction

In this section, we shall first establish the **fundamental law of trigonometry** before discussing the **Trigonometric Identities**. For this we should know the formula to find the distance between two points in a plane.

10.1.1 The Law of Cosine

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points. If " d " denotes the distance between them,

then, $d = |\overline{PQ}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e., square root o f the sum of square the difference of x-coordinates and square the difference o f y-coordinates.

Example 1: Find distance between the following points:

i) $A(3,8)$, $B(5,6)$

ii) $P(\cos x, \cos y)$, $Q(\sin x, \sin y)$

Solution:

i) Distance $= |\overline{AB}| = \sqrt{(3-5)^2 + (8-6)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$$= \sqrt{(5-3)^2 + (6-8)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

ii) Distance $= \sqrt{(\cos x - \sin x)^2 + (\cos y - \sin y)^2}$

$$= \sqrt{\cos^2 x + \sin^2 x - 2\cos x \sin x + \cos^2 y + \sin^2 y - 2\cos y \sin y}$$

$$= \sqrt{2 \quad 2\cos x \sin x \quad 2\cos y \sin y}$$

$$= \sqrt{2 \quad 2(\cos x \sin x \quad \cos y \sin y}$$

10.1.2 Fundamental Law of trigonometry

Let α and β any two angles (real numbers), then

$$\boxed{\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta}$$

which is called the **Fundamental Law of Trigonometry**.

Proof: For our convenience, let us assume that $\alpha > \beta > 0$.

Consider a unit circle with centre at origin O.

Let terminal sides o f angles α and β cut the unit circle at A and B respectively.

Evidently $\angle AOB = \alpha - \beta$

Take a point C on the unit circle so that

$$\angle XOC = \angle m \quad AOB \quad \alpha \quad \beta.$$

Join A,B and C,D.

Now angles α β and $\alpha - \beta$ are in standard position.

$\therefore$ The coordinates o f A are $(\cos\alpha, \sin\alpha)$

the coordinates o f B are $(\cos\beta, \sin\beta)$

the coordinates o f C are $(\cos\overline{\alpha-\beta}, \sin\overline{\alpha-\beta})$

and the coordinates o f D are (1, 0).

Now ΔAOB and ΔCOD are congruent. [(SAS) theorem]

$\therefore \quad |\overline{AB}| = |\overline{CD}|$

$\Rightarrow \quad |\overline{AB}|^2 = |\overline{CD}|^2$

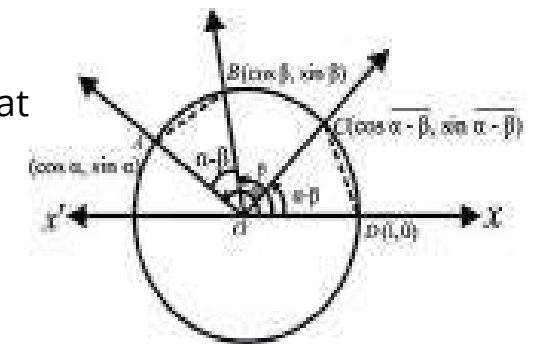
Using the distance formula, we have:

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = [(\cos(\alpha-\beta) - 1]^2 + [\sin(\alpha-\beta) - 0]^2$$

$$\Rightarrow \quad \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta$$

$$= \cos^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta) + \sin^2(\alpha-\beta)$$

$$\Rightarrow \quad 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha-\beta)$$

Hence $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Note: Although we have proved this law for α > β > 0. it is true for all values of α and β

Suppose we know the values of sin and cos of two angles α and β. we can find cos (α – β) using this law as explained in the following example:

Example 1: Find the value of $\cos\frac{\pi}{12}$.

Solution: **As** $\frac{\pi}{12} = 15^\circ = 45^\circ - 30^\circ = \frac{\pi}{4} - \frac{\pi}{6}$

$$\therefore \quad \cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

10.2 Deductions from Fundamental Law

1) We know that:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Putting $\alpha = \frac{\pi}{2}$ in it, we get

$$\cos\left(\frac{\pi}{2}-\beta\right) = \cos\frac{\pi}{2}\cos\beta + \sin\frac{\pi}{2}\sin\beta$$

$$\Rightarrow \cos\left(\frac{\pi}{2}-\beta\right) = 0.\cos\beta + 1.\sin\beta \qquad \left(\because \cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1\right)$$

$$\therefore \boxed{\cos\left(\frac{\pi}{2}-\beta\right) = \sin\beta}$$

2) We know that:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Putting $\beta = -\frac{\pi}{2}$ in it, we get

$$\cos\left[\alpha - \left(-\frac{\pi}{2}\right)\right] = \cos\alpha.\cos\left(-\frac{\pi}{2}\right) + \sin\alpha\sin\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \cos\left(\alpha+\frac{\pi}{2}\right) = \cos\alpha.0 + \sin\alpha.(-1) \qquad \begin{cases}\sin\left(-\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1\\ \cos\left(-\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0\end{cases}$$

$$\therefore \boxed{\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha} \qquad \text{(ii)}$$

3) We known that

$$\cos\left(\frac{\pi}{2}-\beta\right) = \sin\beta \qquad \text{[(i) above]}$$

Putting $\beta = \frac{\pi}{2}+\alpha$ in it, we get

$$\cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2}+\alpha\right)\right] = \sin\left(\frac{\pi}{2}+\alpha\right)$$

$$\Rightarrow \cos(-\alpha) = \sin\left(\frac{\pi}{2}+\alpha\right)$$

$$\Rightarrow \cos\alpha = \sin\left(\frac{\pi}{2}+\alpha\right) \qquad \{\because \cos(-\alpha) = \cos\alpha\}$$

$$\therefore \boxed{\sin\left(\frac{\pi}{2}+\alpha\right) = \cos\alpha} \qquad \text{(iii)}$$

4) We known that

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

replacing β by $-\beta$ we get

$$\cos[\alpha-(-\beta)] = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

$$\{\because \cos(-\beta) = \cos\beta, \sin(-\beta) = -\sin\beta\}$$

7 $\Rightarrow \boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$

5) We known that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

replacing α by $\frac{\pi}{2}+\alpha$ we get

$$\cos\left[\left(\frac{\pi}{2}+\alpha\right)+\beta\right] = \cos\left(\frac{\pi}{2}+\alpha\right)\cos\beta - \sin\left(\frac{\pi}{2}+\alpha\right)\sin\beta$$

$$\Rightarrow \quad \cos\left[\frac{\pi}{2}+(\alpha+\beta)\right] = -\sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\Rightarrow \quad -\sin(\alpha+\beta) = -[\sin\alpha\cos\beta + \cos\alpha\sin\beta]$$

$$\therefore \quad \boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

6) We known that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \text{[from (v) above]}$$

replacing β by $-\beta$ we get

$$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

$$\because \begin{cases} \sin(-\beta) = -\sin\beta \\ \cos(-\beta) = \cos\beta \end{cases}$$

$$\therefore \quad \boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta} \qquad \text{(vi)}$$

7) We known that

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha \,.\, \sin\beta$$

Let $\alpha = 2\pi$ and $\beta = \theta$

$$\therefore \quad \cos(2\pi-\theta) = \cos 2\pi \,.\, \cos\theta \quad \sin 2\pi \sin\theta$$

$$= 1 \,.\, \cos\theta + 0 \,.\, \sin\theta \qquad \because \begin{cases} \cos 2\pi = 1 \\ \sin 2\pi = 0 \end{cases}$$

$$= \cos\theta$$

8) We known that $\sin(\alpha-\beta) = \sin\alpha \,.\, \cos\beta - \cos\alpha \,.\, \sin\beta$

$$\therefore \quad \sin(2\pi-\theta) = \sin 2\pi \,.\, \cos\theta - \cos 2\pi \sin\theta$$

$$= 0 \,.\, \cos\theta - 1 \,.\, \sin\theta \qquad \because \begin{cases} \sin 2\pi = 0 \\ \cos 2\pi = 1 \end{cases}$$

$$= -\sin\theta \qquad \text{(viii)}$$

9) $$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \qquad \left[\begin{array}{c}\text{Dividing} \\ \text{neumerator and} \\ \text{denuminator} \\ -\cos\alpha\cos\beta\end{array}\right]$$

$$\therefore \quad \boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}} \qquad \text{(ix)}$$

10) $$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \qquad \left[\begin{array}{c}\text{Dividing} \\ \text{neumerator and} \\ \text{denuminator}\end{array}\right]$$

$$\therefore \quad \boxed{\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}} \qquad \text{(x)}$$

10.3 Trigonometric Ratios of Allied Angles

The angles associated with basic angles of measure θ to a right angle or its multiples are called **allied angles**. So, the angles of measure 90° ± θ , 180° ± θ, 270° ± θ, 360° ± θ, are known as **allied angles**.

Using fundamental law, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and its deductions, we derive the following identities:

$$\begin{cases}\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta, & \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta, \quad \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta\\ \sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta, & \cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta, \quad \tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta\end{cases}$$

$$\begin{cases}\sin(\pi-\theta)=\sin\theta, & \cos(\pi-\theta)=-\cos\theta, \quad \tan(\pi-\theta)=-\tan\theta\\ \sin(\pi+\theta)=-\sin\theta, & \cos(\pi+\theta)=-\cos\theta, \quad \tan(\pi+\theta)=\tan\theta\end{cases}$$

$$\begin{cases}\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta, & \cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta, \quad \tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta\\ \sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta, & \cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta, \quad \tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta\end{cases}$$

$$\begin{cases}\sin(2\pi-\theta)=-\sin\theta, & \cos(2\pi-\theta)=\cos\theta, \quad \tan(2\pi-\theta)=-\tan\theta\\ \sin(2\pi+\theta)=\sin\theta, & \cos(2\pi+\theta)=\cos\theta, \quad \tan(2\pi+\theta)=\tan\theta\end{cases}$$

Note: The above results also apply to the reciprocals of sine, cosine and tangent. These results are to be applied frequently in the study of trigonometry, and they can be remembered by using the following device:

1) If θ is added to or subtracted from **odd multiple** of right angle, the trigonometric ratios change into **co-ratios** and vice versa.

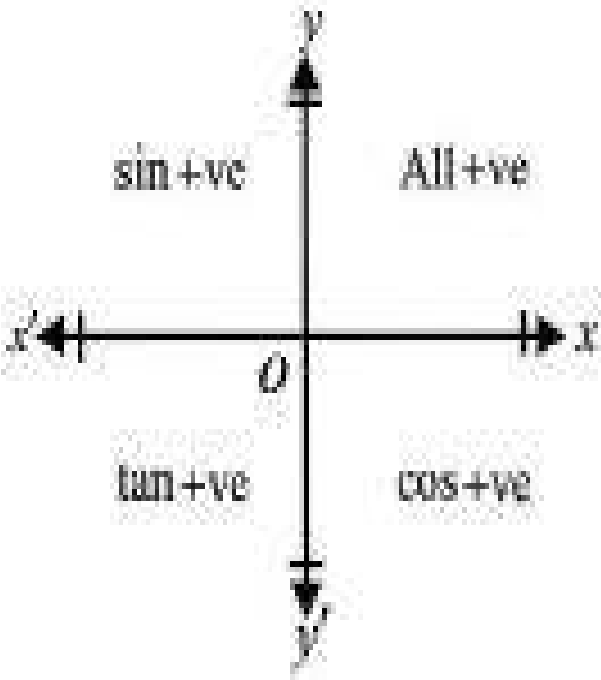
i.e, $\sin \rightleftarrows \cos$, $\tan \rightleftarrows \cot$, $\sec \rightleftarrows \operatorname{cosec}$

e.g. $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ and $\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$

2) If θ is added to or subtracted from an even multiple of $\frac{\pi}{2}$ the trigonometric ratios shall remain the same.

3) So far as the sign of the results is concerned, it is determined by the quadrant in which the terminal arm of the angle lies.

e.g. $\sin(\pi-\theta)=\sin\theta$, $\tan(\pi+\theta)=\tan\theta$, $\cos(2\pi-\theta)=\cos\theta$

Measure of the angle	Quad.
$\frac{\pi}{2}-\theta$	I
$\frac{\pi}{2}+\theta$ or $\pi-\theta$	II
$\pi+\theta$ or $\frac{3\pi}{2}-\theta$	III
$\frac{3\pi}{2}+\theta$ or $2\pi-\theta$	Iv

a) In $\sin\left(\frac{\pi}{2}-\theta\right)$, $\sin\left(\frac{\pi}{2}+\theta\right)$, $\sin\left(\frac{3\pi}{2}-\theta\right)$ and $\sin\left(\frac{3\pi}{2}+\theta\right)$

odd multiples of $\frac{\pi}{2}$ are involved.

$\therefore$ sin will change into cos.

Moreover, the angle of measure

i) $\left(\frac{\pi}{2}-\theta\right)$ will have terminal side in Quad.I,

So $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$;

ii) $\left(\frac{\pi}{2}+\theta\right)$ will have terminal side in Quad.II,

So $\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$;

iii) $\left(\frac{3\pi}{2}-\theta\right)$ will have terminal side in Quad.III,

So $\sin\left(\frac{3\pi}{2}-\theta\right) = \quad \cos\theta;$

iv) $\left(\frac{3\pi}{2}+\theta\right)$ will have terminal side in Quad.IV,

So $\sin\left(\frac{3\pi}{2}+\theta\right) = \quad \cos\theta.$

b) In $\cos(\pi-\theta)$, $\cos(\pi+\theta)$, $\cos(2\pi-\theta)$ and $\cos(2\pi+\theta)$ even multiples of $\frac{\pi}{2}$ are involved.

$\therefore$ cos will remain as cos.

Moreover, the angle of measure

i) $(\pi-\theta)$ will have terminal side in Quad. II,

$\therefore \cos(\pi-\theta) = -\cos\theta;$

ii) $(\pi+\theta)$ will have terminal side in Quad. III,

$\therefore \cos(\pi+\theta) = -\cos\theta;$

iii) $(2\pi-\theta)$ will have terminal side in Quad. IV:

$\therefore \cos(2\pi-\theta) = \cos\theta;$

iv) $(2\pi+\theta)$ will have terminal side in Quad. I

$\therefore \cos(2\pi+\theta) = \cos\theta.$

Example 2: Without using the tables, write down the values of:

i) cos 315° ii) sin 540° iii) tan (–135°) iv) sec(–300°)

Solution:

i) $\cos 315^\circ = \cos(270+45)^\circ = \cos(3\times 90+45)^\circ = +\sin 45^\circ = \frac{1}{\sqrt{2}}$

ii) $\sin 540^\circ = \sin(540+0)^\circ = \sin(6\times 90+0)^\circ = +\sin 0 = 0$

iii) $\tan(-135^\circ) = \tan 135^\circ \quad -\tan(180 \times 45)^\circ - \tan(2 \quad 90 \quad 45)^\circ \quad (\tan 45^\circ) \quad 1$

iv) $\sec(-300^\circ) = \sec 300^\circ = \sec(360-60)^\circ = \sec(4\times 90-60)^\circ = \sec 60^\circ = 2$

Example 3: Simplify:

$$\frac{\sin(360^\circ-\theta)\cos(180^\circ-\theta)\tan(180^\circ+\theta)}{\sin(90^\circ+\theta)\cos(90^\circ-\theta)\tan(360^\circ+\theta)}$$

Solution: $\because$ $\begin{cases} \sin(360^\circ-\theta) = -\sin\theta, & \cos(180^\circ \quad \theta) = \cos\theta \\ \tan(180^\circ+\theta) = \tan\theta, & \sin(90^\circ \quad \theta) \quad \cos\theta \\ \cos(90^\circ-\theta) = \sin\theta, & \tan(360^\circ \quad \theta) \quad \tan\theta \end{cases}$

$$\therefore \frac{\sin(360^\circ-\theta)\cos(180^\circ-\theta)\tan(180^\circ+\theta)}{\cos\theta . \sin\theta . \tan\theta} = \frac{(-\sin\theta)(-\cos\theta)\tan\theta}{\cos\theta . \sin\theta . \tan\theta} \quad 1$$

Exercise 10.1

1. Without using the tables, find the values of:

i) $\sin(-780^\circ)$ ii) $\cot(-855^\circ)$ iii) $\csc(2040^\circ)$

iv) $\sec(-960^\circ)$ v) $\tan(1110^\circ)$ vi) $\sin(-300^\circ)$

2. Express each of the following as a trigonometric function of an angle of positive degree measure of less than 45°.

i) $\sin 196^\circ$ ii) $\cos 147^\circ$ iii) $\sin 319^\circ$

iv) $\cos 254^\circ$ v) $\tan 294^\circ$ vi) $\cos 728^\circ$

vii) $\sin(-625^\circ)$ viii) $\cos(-435^\circ)$ ix) $\sin 150^\circ$

3. Prove the following:

i) $\sin(180^\circ+\alpha)\sin(90-\alpha) = \quad \sin\alpha\cos\alpha$

ii) $\sin 780^\circ \sin 480^\circ + \cos 120^\circ \sin 30^\circ = \frac{1}{2}$

iii) $\cos 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ = 0$

iv) $\cos 330^\circ \sin 600^\circ + \cos 120^\circ \sin 150^\circ = \quad 1$

4. Prove that:

i) $$\frac{\sin^2(\pi+\theta)\tan\left(\frac{3\pi}{2}+\theta\right)}{\cot^2\left(\frac{3\pi}{2}-\theta\right)\cos^2(\pi-\theta)\operatorname{coses}(2\pi-\theta)} = \cos\theta$$

ii) $$\frac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)} = -1$$

5. If α, β, γ are the angles of a triangle ABC, then prove that

i) $\sin(\alpha+\beta) = \sin\gamma$ ii) $\cos\left(\frac{\alpha+\beta}{2}\right) = \sin\frac{\gamma}{2}$

iii) $\cos(\alpha+\beta) -= \cos\gamma$ iv) $\tan(\alpha+\beta)+\tan\gamma = 0.$

10.4 Further Application of Basic Identities

Example 1: Prove that

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha-\sin^2\beta \qquad \text{(i)}$$
$$= \cos^2\beta-\cos^2\alpha \qquad \text{(ii)}$$

Solution: L.H.S. $= \sin(\alpha+\beta)\sin(\alpha-\beta)$

$= (\sin\alpha\cos\beta \quad \cos\alpha\sin\beta)(\sin\alpha\cos\beta \quad \cos\alpha\sin\beta)$

$= -\sin^2\alpha\cos^2\beta \quad \cos^2\alpha\sin^2\beta$

$= \sin^2\alpha(1-\sin^2\beta)-(1-\sin^2\alpha)\sin^2\beta$

$= \sin^2\alpha-\sin^2\alpha\sin^2\beta-\sin^2\beta+\sin^2\alpha\sin^2\beta$

$= \sin^2\alpha-\sin^2\beta$ (i)

$= (1-\cos^2\alpha)-(1-\cos^2\beta)$

$= 1-\cos^2\alpha-1+\cos^2\beta$

$= \cos^2\beta-\cos^2\alpha$ (ii)

Example 2: Without using tables, find the values of all trigonometric functions of 75°.

Solution: As $75^\circ = 45^\circ+30^\circ$

$$\sin 75^\circ = \sin(45^\circ+30^\circ) = \sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos 75^\circ = \cos(45^\circ+30^\circ) = \cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\tan 75^\circ = \tan(45^\circ+30^\circ) = \frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-1.\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\cot 75^\circ = \frac{1}{\tan 75^\circ} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\operatorname{coses} 75^\circ = \frac{1}{\sin 75^\circ} = \frac{2\sqrt{2}}{\sqrt{3}+1} \quad \text{and} \quad \sec 75^\circ = \frac{1}{\cos 75^\circ} = \frac{2\sqrt{2}}{\sqrt{3}-1}$$

Example 3: Prove that: $\dfrac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ} = \tan 56^\circ.$

Solution: Consider

$$\text{R.H.S.} = \tan 56^\circ = \tan(45^\circ+11^\circ) = \frac{\tan 45^\circ+\tan 11^\circ}{1-\tan 45^\circ\tan 11^\circ}$$

$$= \frac{1+\tan 11^\circ}{1-\tan 11^\circ} = \frac{1+\frac{\sin 11\circ}{\cos 11^\circ}}{1-\frac{\sin 11^\circ}{\cos 11^\circ}} = \frac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ} = \text{L.H.S.}$$

Hence $\dfrac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \tan 56^\circ.$

Example 4: If $\cos\alpha = -\dfrac{24}{25}$, $\tan\beta = \dfrac{9}{40}$ the terminal side of the angle of measure α is in the II quadrant and that of β is in the III quadrant, find the values of:

i) $\sin(\alpha+\beta)$ ii) $\cos(\alpha+\beta)$

In which quadrant does the terminal side of the angle of measure $(\alpha+\beta)$ lie?

Solution: We know that $\sin^2\alpha + \cos^2\alpha = 1$

$$\therefore \quad \sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\sqrt{1-\frac{576}{625}} = \pm\sqrt{\frac{49}{625}} = \pm\frac{7}{25}$$

As the terminal side o f the angle of measure of α is in the II quadrant, where sin α is positive.

$$\therefore \quad \sin\alpha = \frac{7}{25}$$

Now $\sec\beta = \pm\sqrt{1+\tan^2\beta} = \pm\sqrt{1+\dfrac{81}{1600}} = \pm\dfrac{41}{40}$

As the terminal side of the angle of measure of β in the III quadrant, so sec β is negative

$$\therefore \quad \sec\beta = -\frac{41}{40} \quad \text{and} \quad \cos\beta = -\frac{40}{41}$$

$$\sin\beta = \pm\sqrt{1-\cos^2\beta} = \pm\sqrt{1-\frac{1600}{1681}} = \pm\frac{9}{41}$$

As the terminal arm of the angle of measure β is in the III quadrant, so sin β is negative

$$\therefore \quad \sin\beta = -\frac{9}{41}$$

$$\therefore \quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \left(\frac{7}{25}\right)\left(-\frac{40}{41}\right) + \left(-\frac{24}{25}\right)\left(-\frac{9}{41}\right)$$

$$= \frac{-280+216}{1025} = -\frac{64}{1025}$$

and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{24}{25}\right)\left(-\frac{40}{41}\right) - \left(\frac{7}{25}\right)\left(-\frac{9}{41}\right)$$

$$= \frac{960+63}{1025}$$

$$= \frac{1023}{1025}$$

$\because$ $\sin(\alpha+\beta)$ is $-$ve and $\cos(\alpha+\beta)$ is $+$ve

$\therefore$ The terminal arm of the angle of measure $(\alpha+\beta)$ is in the IV quadrant.

Example 5: If α, β, γ are the angles of ΔABC prove that:

i) $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

ii) $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} + \tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2} + \tan\dfrac{\gamma}{2}\tan\dfrac{\alpha}{2} = 1$

Solution: As α, β, γ are the angles of ΔABC

$$\therefore \quad \alpha+\beta+\gamma = 180^\circ$$

i) $\alpha+\beta = 180^\circ - \gamma$

$$\therefore \quad \tan(\alpha+\beta) = \tan(180^\circ-\gamma)$$

$$\Rightarrow \quad \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \tan\gamma$$

$$\Rightarrow \tan\alpha + \tan\beta = -\tan\gamma + \tan\alpha\tan\beta\tan\gamma$$

$$\therefore \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$$

ii) As $\alpha + \beta + \gamma = 180^\circ \quad \Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90^\circ$

So $\frac{\alpha}{2} + \frac{\beta}{2} = 90^\circ - \frac{\gamma}{2}$

$$\therefore \tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(90^\circ - \frac{\gamma}{2}\right)$$

$$\Rightarrow \frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} == \cot\frac{\gamma}{2} \quad \frac{1}{\tan\frac{\gamma}{2}}$$

$$\Rightarrow \tan\frac{\alpha}{2}\tan\frac{\gamma}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} = 1 \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$\therefore \tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$$

Example 6: Express $3\sin\theta + 4\cos\theta$ in the form $r\sin(\theta + \phi)$, where the terminal side of the angle of measure ϕ is in the I quadrant.

Solution: Let $3 == r\cos\phi$ and $4 \quad r\sin\phi$

$$\therefore 3^2 + 4^2 = r^2\cos^2\phi + r^2\sin^2\phi$$

$$\Rightarrow 9 + 16 = r^2(\cos^2\phi + \sin^2\phi)$$
$$\Rightarrow 25 = r^2$$
$$\Rightarrow 5 = r$$
$$\Rightarrow r = 5$$

$$\frac{4}{3} = \frac{r\sin\phi}{r\cos\phi}$$
$$\Rightarrow \frac{4}{3} = \tan\phi$$
$$\therefore \tan\phi = \frac{4}{3}$$

$$\therefore 3\sin\theta + 4\cos\theta = r\cos\phi\sin\theta + r\sin\phi\cos\theta$$
$$=+ r(\sin\theta\cos\phi \quad \cos\theta\sin\phi)$$

$$= r\sin(\theta + \phi)$$

Where $r == 5$ and $\phi \quad \tan^{-1}\frac{4}{3}$

Exercise 10.2

1. Prove that

i) $\sin(180^\circ + \theta) = -\sin\theta$ ii) $\cos(180^\circ + \theta) = -\cos\theta$

iii) $\tan(270^\circ - \theta) = \cot\theta$ iv) $\cos(\theta - 180^\circ) = \cos\theta$

v) $\cos(270^\circ + \theta) = \sin\theta$ vi) $\sin(\theta + 270^\circ) = \cos\theta$

vii) $\tan(180^\circ + \theta) = \tan\theta$ viii) $\cos(360^\circ - \theta) = \cos\theta$

2. Find the values o f the following:

i) $\sin 15^\circ$ ii) $\cos 15^\circ$ iii) $\tan 15^\circ$

iv) $\sin 105^\circ$ v) $\cos 105^\circ$ vi) $\tan 105^\circ$

(**Hint:** $15^\circ = (45^\circ \quad 30^\circ)$ and $105^\circ \quad (60^\circ \quad 45^\circ)$)

3. Prove that:

i) $\sin(45^\circ + \alpha) = \frac{1}{\sqrt{2}}(\sin\alpha + \cos\alpha)$

ii) $\cos(\alpha + 45^\circ) = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)$

4. Prove that:

i) $\tan(45^\circ + A)\tan(45^\circ - A) = 1$

ii) $\tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right) = 0$

iii) $\sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

iv) $\dfrac{\sin\theta - \cos\theta\tan\frac{\theta}{2}}{\cos\theta + \sin\theta\tan\frac{\theta}{2}} = \tan\frac{\theta}{2}$

v) $\dfrac{1-\tan\theta\tan\phi}{1+\tan\theta\tan\phi} = \dfrac{\cos(\theta+\phi)}{\cos(\theta-\phi)}$

5. Show that: $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$

6. Show that: $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$

7. Show that:

i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

ii) $\cot(\alpha-\beta) = \dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

8. If $\sin\alpha = \frac{4}{5}$ and $\cos\beta \ \frac{40}{41}$, where $0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$.

Show that $\sin(\alpha-\beta) = \dfrac{133}{205}$.

9. If $\sin\alpha = \frac{4}{5} <$ and $\sin\beta = \frac{12}{13} <$ where $\frac{\pi}{2} \ \alpha \ \pi$ and $\frac{\pi}{2} \ \beta \ \pi$. Find

i) $\sin(\alpha+\beta)$ ii) $\cos(\alpha+\beta)$ iii) $\tan(\alpha+\beta)$

iv) $\sin(\alpha-\beta)$ v) $\cos(\alpha-\beta)$ vi) $\tan(\alpha-\beta)$.

In which quadrants do the terminal sides of the angles of measures $(\alpha+\beta)$ and $(\alpha-\beta)$ lie?

10. Find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, given that

i) $\tan\alpha = \frac{3}{4}$, $\cos\beta = \frac{5}{13}$ and neither the terminal side o f the angle of measure α nor that of β is in the I quadrant.

ii) $\tan\alpha = \frac{15}{8}$ and $\sin\beta \ \frac{7}{25}$ and neither the terminal side of the angle of measure α nor that of β is in the IV quadrant.

11. Prove that $\dfrac{\cos 8^\circ \quad \sin 8^\circ}{\cos 8^\circ \quad \sin 8^\circ} \quad \tan 37$

12. If α,β,γ are the angles of a triangle *ABC*, show that $\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

13. If $\alpha+\beta+\gamma = \ 180^\circ$, show that

$\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha = 1$

14. Express the following in the form $r\sin(\theta+\phi)$ or $r\sin(\theta-\phi)$, where terminal sides of the angles of measures θ and ϕ are in the first quadrant:

i) $12\sin\theta+5\cos\theta$ ii) $3\sin\theta-4\cos\theta$ iii) $\sin\theta-\cos\theta$

iv) $5\sin\theta-4\cos\theta$ v) $\sin\theta+\cos\theta$. vi) $3\sin\theta-5\cos\theta$

10.5 Double angle Identities

We have discovered the following results:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$$

and $$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

We can use them to obtain the double angle identities as follows:

i) Put $\beta=\alpha$ in $\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\therefore \quad \sin(\alpha+\alpha) = \sin\alpha\cos\alpha+\cos\alpha\sin\alpha$

Hence $\sin 2\alpha = 2\sin\alpha\cos\alpha$

ii) Put $\beta=\alpha$ in $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

Hence $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\because \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\therefore \quad \cos 2\alpha = \cos^2\alpha - (1-\cos^2\alpha) \qquad (\because \sin^2\alpha = 1-\cos^2\alpha)$

$= \cos^2\alpha - 1 + \cos^2\alpha$

$\therefore \quad \cos 2\alpha = 2\cos^2\alpha - 1$

$\because \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\therefore \quad \cos 2\alpha = (1-\sin^2\alpha) - \sin^2\alpha \qquad (\because \cos^2\alpha = 1-\sin^2\alpha)$

$\therefore \quad \cos 2\alpha = 1 - 2\sin^2\alpha$

iii) Put $\beta=\alpha$ in $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\tan(\alpha+\alpha) = \dfrac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$

$\therefore \quad \tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

10.6 Half angle Identities

The formulas proved above can also be written in the form of half angle identities, in the following way:

i) $\because \cos\alpha = 2\cos^2\dfrac{\alpha}{2} - 1 \quad \Rightarrow \cos^2\dfrac{\alpha}{2} = \dfrac{1+\cos\alpha}{2}$

$\Rightarrow \cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$

ii) $\because \cos\alpha = 1 - 2\sin^2\dfrac{\alpha}{2} \Rightarrow \sin^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{2}$

$\Rightarrow \sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$

iii) $\tan\dfrac{\alpha}{2} = \dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \pm\dfrac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}}$

$\Rightarrow \tan\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$

10.7 Triple angle Identities

i) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

ii) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

iii) $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$

Proof:

i) $\sin 3\alpha = \sin(2\alpha+\alpha)$

$= \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha$

$= 2\sin\alpha\cos\alpha\cos\alpha + (1-2\sin^2\alpha)\sin\alpha$

$= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha$

$= 2\sin\alpha(1-\sin^2\alpha) + \sin\alpha - 2\sin^3\alpha$

$$= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha$$

$$\therefore \quad \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

ii) $\cos 3\alpha = \cos(2\alpha + \alpha)$

$$= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha$$
$$= (2\cos^2\alpha - 1)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2(1-\cos^2\alpha)\cos\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha$$

$$\therefore \quad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

iii) $\tan 3\alpha = \tan(2\alpha + \alpha)$

$$= \frac{\tan 2\alpha + \tan\alpha}{1 - \tan 2\alpha\tan\alpha}$$

$$= \frac{\frac{2\tan\alpha}{1-\tan^2\alpha} + \tan\alpha}{1 - \frac{2\tan\alpha}{1-\tan^2\alpha}\cdot\tan\alpha} = \frac{2\tan\alpha + \tan\alpha - \tan^3\alpha}{1 - \tan^2\alpha - 2\tan^2\alpha}$$

$$\therefore \quad \tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

Example 1: Prove that $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A$

Solution: L.H.S $= \dfrac{\sin A + 2\sin A\cos A}{1 + \cos A + 2\cos^2 A - 1} = \dfrac{\sin A(1 + 2\cos A)}{\cos A(1 + 2\cos A)}$

$$= \frac{\sin A}{\cos A} = \tan A = \text{R.H.S}$$

Hence $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A.$

Example 2: Show that

i) $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$ ii) $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Solution: i) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\sin\theta\cos\theta}{1} = \dfrac{2\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta}$

$$= \frac{\frac{2\sin\theta\cos\theta}{\cos^2\theta}}{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}} = \frac{2\frac{\sin\theta}{\cos\theta}}{\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}}$$

$$\therefore \quad \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = \dfrac{\cos^2\theta - \sin^2\theta}{1} = \dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$

$$= \frac{\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}} = \frac{\frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}}$$

$$\therefore \quad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

Example 3: Reduce $\cos^4\theta$ to an expression involving only function of multiples of θ, raised to the first power.

Solution: We know that:

$$2\cos^2\theta = 1 + \cos 2\theta \quad \Rightarrow \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\therefore \quad \cos^4\theta = (\cos^2\theta)^2 = \left[\frac{1+\cos 2\theta}{2}\right]^2$$

$$= \frac{1+2\cos 2\theta+\cos^2 2\theta}{4}$$

$$= \frac{1}{4}[1+2\cos 2\theta+\cos^2 2\theta]$$

$$= \frac{1}{4}\left[1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right]$$

$$= \frac{1}{4\times 2}[2+4\cos 2\theta+1+\cos 4\theta]$$

$$= \frac{1}{8}[3+4\cos 2\theta+\cos 4\theta]$$

Exercise 10.3

1. Find the values of $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$, when:

i) $\sin\alpha = \frac{12}{13}$ ii) $\cos\alpha = \frac{3}{5}$, where $0<\alpha<\frac{\pi}{2}$

Prove the following identities:

2. $\cot\alpha - \tan\alpha = 2\cot 2\alpha$

3. $\frac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$

4. $\frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}$

5. $\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha} = \sec 2\alpha - \tan 2\alpha$

6. $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \frac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}}$

7. $\frac{\text{coses}\,\theta+\text{coses}\,2\theta}{\sec\theta} = \cot\frac{\theta}{2}$

8. $1+\tan\alpha\tan 2\alpha = \sec 2\alpha$

9. $\frac{2\sin\theta\sin 2\theta}{\cos\theta+\cos 3\theta} = \tan 2\theta\tan\theta$

10. $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$

11. $\frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta$

12. $\frac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}} = \sec\theta$

13. $\frac{\sin 3\theta}{\cos\theta} + \frac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$

14. Reduce $\sin^4\theta$ to an expression involving only function of multiples of θ, raised to the first power.

15. Find the values of $\sin\theta$ and $\cos\theta$ without using table or calculator, when θ is

i) 18° ii) 36° iii) 54° iv) 72°

Hence prove that: $\cos 36^\circ\cos 72^\circ\cos 108^\circ\cos 144^\circ = \frac{1}{16}$

Hint :

Let $\theta == 18^\circ$	Let $\theta \quad 36^\circ$
$5\theta == 90^\circ$	$5\theta \quad 180^\circ$
$(3\theta+2\theta) = 90^\circ + \quad =$	$3\theta \quad 2\theta \quad 180^\circ$
$3\theta = 90^\circ \quad 2\theta$	$-3\theta \quad 180^\circ \quad 2\theta =$
$\sin 3\theta = \sin(90^\circ \quad 2\theta)$	$\sin 3\theta \quad -\sin(180^\circ \quad 2\theta)$
etc.	etc.

10.8. Sum, Difference and Product of Sines and Cosines

We know that:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta \quad \text{(i)}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta \quad \text{(ii)}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta \quad \text{(iii)}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta \quad \text{(iv)}$$

Adding (i) and (ii) we get

$$\sin(\alpha+\beta)+\sin(\alpha-\beta) = 2\sin\alpha\cos\beta \quad \text{(v)}$$

Subtracting (ii) from (i) we get

$$\sin(\alpha+\beta)-\sin(\alpha-\beta) = 2\cos\alpha\sin\beta \qquad \text{(vi)}$$

Adding (iii) and (iv) we get

$$\cos(\alpha+\beta)+\cos(\alpha-\beta) = 2\cos\alpha\cos\beta \qquad \text{(vii)}$$

Subtracting (iv) from (iii) we get

$$\cos(\alpha+\beta)-\cos(\alpha-\beta) = -2\sin\alpha\sin\beta \qquad \text{(viii)}$$

So we get four identities as:

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta)-\sin(\alpha-\beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$$

$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta)-\cos(\alpha-\beta)$$

Now putting $\alpha+\beta = P$ and $\alpha-\beta = Q$, we get

$$\alpha == \frac{P+Q}{2} \quad \text{and} \quad \beta \quad \frac{P-Q}{2}$$

$$\sin P+\sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\sin P-\sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

$$\cos P+\cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\cos P-\cos Q = \ 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

Example 1: Express $2\sin 7\theta\cos 3\theta$ as a sum or difference.

Solution: $2\sin 7\theta\cos 3\theta = \sin(7\theta+3\theta)+\sin(7\theta-3\theta)$

$$= \sin 10\theta + \sin 4\theta$$

Example 2: Prove without using tables / calculator, that

$$\sin 19^\circ\cos 11^\circ + \sin 71^\circ\sin 11^\circ = \frac{1}{2}$$

Solution: L.H.S. $=+ \sin 19^\circ\cos 11^\circ \quad \sin 71^\circ\sin 11^\circ$

$$=+\frac{1}{2}[2\sin 19^\circ\cos 11^\circ \quad 2\sin 71^\circ\sin 11^\circ]$$

$$= \frac{1}{2}[\{\sin(19^\circ+11^\circ)+\sin(19^\circ-11^\circ)\}-\{\cos(71^\circ+11^\circ)-\cos(71^\circ-11^\circ)\}]$$

$$= \frac{1}{2}[\sin 30^\circ+\sin 8^\circ-\cos 82^\circ+\cos 60^\circ]$$

$$= \frac{1}{2}\left[\frac{1}{2}+\sin 8^\circ-\cos(90^\circ-8^\circ)+\frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}+\sin 8^\circ-\sin 8^\circ+\frac{1}{2}\right] \quad (\because \cos 82^\circ = \cos(90^\circ-8^\circ) = \sin 8^\circ)$$

$$= \frac{1}{2}\left[\frac{1}{2}+\frac{1}{2}\right]$$

$$= \frac{1}{2}$$

= R.H.S.

Hence $\sin 19^\circ\cos 11^\circ + \sin 71^\circ + \sin 11^\circ = \frac{1}{2}$

Example 3: Express $\sin 5x+\sin 7x$ as a product.

Solution: $\sin 5x+\sin 7x = 2\sin\frac{5x+7x}{2}\cos\frac{5x-7x}{2} = 2\sin 6x\cos(\ x)$

$$= 2\sin 6x\cos x \qquad (\because \cos(\ \theta) \quad \cos\theta)$$

Example 4: Express $\cos A+\cos 3A+\cos 5A+\cos 7A$ as a product.

Solution: $\cos A+\cos 3A+\cos 5A+\cos 7A$

$=(\cos 3A+\cos A)+(\cos 7A+\cos 5A)$

$=2\cos\frac{3A+A}{2}\cos\frac{3A-A}{2} \quad 2\cos\frac{7A+A}{2}\cos\frac{7A-5A}{2}$

$=2\cos 2A\cos A \quad 2\cos 6A\cos A$

$=2\cos A(\cos 6A \quad \cos 2A)$

$=2\cos A\left[2\cos\frac{6A+2A}{2}\cos\frac{6A-2A}{2}\right]$

$=2\cos A(2\cos 4A\cos 2A) \qquad 4\cos A\cos 2A\cos 4A.$

Example 4: Show that $\cos 20^\circ \; \cos 40^\circ \; \cos 80^\circ = \frac{1}{8}$

Solution: L.H.S. $=\cos 20^\circ \; \cos 40^\circ \; \cos 80^\circ$

$=\frac{1}{4}(4\cos 20^\circ \; \cos 40^\circ \; \cos 80^\circ)$

$=\frac{1}{4}[(2\cos 40^\circ \; \cos 20^\circ) \; .2\cos 80^\circ]$

$=\frac{1}{4}[(\cos 60^\circ+\cos 20^\circ).2\cos 80^\circ]$

$=\frac{1}{4}\left[\left(\frac{1}{2}+\cos 20^\circ\right).2\cos 80^\circ\right]$

$=\frac{1}{4}(\cos 80^\circ+2\cos 80^\circ \; \cos 20^\circ)$

$=\frac{1}{4}(\cos 80^\circ+\cos 100^\circ+\cos 60^\circ)$

$=\frac{1}{4}[\cos 80^\circ+\cos(180^\circ-80^\circ)+\cos 60^\circ]$

$=\frac{1}{4}\left(\cos 80^\circ-\cos 80^\circ+\frac{1}{2}\right)$ $[\because \cos(180-\theta)=-\cos\theta]$

$=\frac{1}{4}\left(\frac{1}{2}\right)=\frac{1}{8}$ R.H.S.

Hence $\cos 20^\circ \; \cos 40^\circ \; \cos 80^\circ = \frac{1}{8}.$

Exercise 10.4

1. Express the following products as sums or differences:

i) $2\sin 3\theta\cos\theta$ ii) $2\cos 5\theta\sin 3\theta$

iii) $\sin 5\theta\cos 2\theta$ iv) $2\sin 7\theta\sin 2\theta$

v) $\cos(x+y)\sin(x-y)$ vi) $\cos(2x+30^\circ)\cos(2x-30^\circ)$

vii) $\sin 12^\circ \sin 46^\circ$ viii) $\sin(x+45^\circ)\sin(x-45^\circ)$

2. Express the following sums or differences as products:

i) $\sin 5\theta+\sin 3\theta$ ii) $\sin 8\theta-\sin 4\theta$

iii) $\cos 6\theta+\cos 3\theta$ iv) $\cos 7\theta-\cos\theta$

v) $\cos 12^\circ+\cos 48^\circ$ vi) $\sin(x+30^\circ)+\sin(x-30^\circ)$

3. Prove the following identities:

i) $\frac{\sin 3x-\sin x}{\cos x-\cos 3x}=\cot 2x$ ii) $\frac{\sin 8x+\sin 2x}{\cos 8x+\cos 2x}=\tan 5x$

iii) $\frac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta}=\tan\frac{\alpha-\beta}{2}\cot\frac{\alpha+\beta}{2}$

4. Prove that:

i) $\cos 20^\circ+\cos 100^\circ+\cos 140^\circ=0$

ii) $\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{2}\cos 2\theta$

iii) $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta} = \tan 4\theta$

5. Prove that:

i) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \dfrac{1}{16}$

ii) $\sin\dfrac{\pi}{9}\sin\dfrac{2\pi}{9}\sin\dfrac{\pi}{3}\sin\dfrac{4\pi}{9} = \dfrac{3}{16}$

iii) $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \dfrac{1}{16}$

Trigonometric Functions and their Graphs

11.1 Introduction

Let us first find domains and ranges of trigonometric functions before drawing their graphs.

11.1.1 Domains and Ranges of Sine and Cosine Functions

We have already defined trigonometric functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$ and $\cot\theta$. We know that if $P(x, y)$ is any point on unit circle with center at the origin O such that $\angle XOP = \theta$ is standard position, then

$\cos\theta = x$ and $\sin\theta = y$

$\Rightarrow$ for any real number θ there is one and only one value of each x and y .i.e., of each $\cos\theta$ and $\sin\theta$.

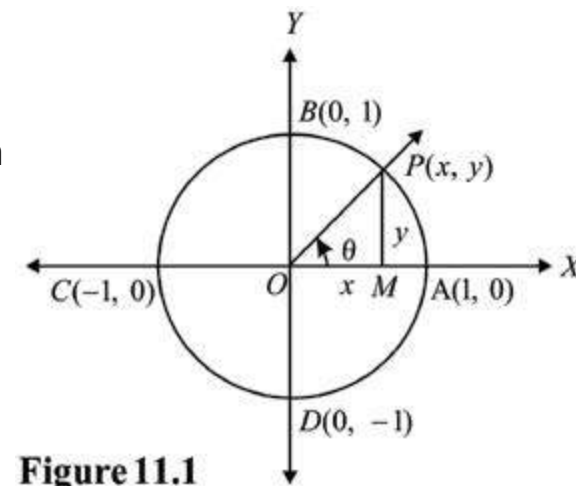



Figure 11.1

Hence $\sin\theta$ and $\cos\theta$ are the functions of θ and their domain is R a set of real numbers. Since $P(x, y)$ is a point on the unit circle with center at the origin O.

$\therefore \quad -1 \le x \le 1 \quad$ and $\quad -1 \le y \le 1$

$\Rightarrow \quad -1 \le \cos\theta \le 1 \quad$ and $\quad -1 \le \sin\theta \le 1$

Thus the range of both the sine and cosine functions is [-1, 1].

11.1.2 Domains and Ranges of Tangent and Cotangent Functions

From figure 11.1

i) $\tan\theta =\neq \frac{y}{x} \quad , \quad x \quad 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OY or OY' (i.e., Y–axis)

$\Rightarrow \quad \theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...$

$\Rightarrow \quad \theta \neq (2n+1)\frac{\pi}{2}, \quad \text{where} \quad n \in Z$

$\therefore$ Domain of tangent function $= R - \{x \mid x = (2n+1)\frac{\pi}{2}, \quad n \in Z\}$

and Range of tangent function = R = set of real numbers.

ii) From figure 11.1

$\cot\theta =\neq \frac{x}{y} \quad , \quad y \quad 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OX or OX' (i.e., X – axis)

$\Rightarrow \quad \theta \neq 0, \pm\pi, \pm 2\pi, ...$

$\Rightarrow \quad \theta \neq n\pi, \quad \text{where} \quad n \in Z$

$\therefore$ Domain of cotangent function $= R - \{x \mid x = n\pi, \quad n \in Z\}$

and Range of cotangent function = R = set of real numbers.

11.1.3 Domain and Range of Secant Function

From figure 11.1

$\sec\theta =\neq \frac{1}{x} \quad , \quad x \quad 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OY or OY' (i.e., Y – axis)

$\Rightarrow \quad \theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...$

$\Rightarrow \quad \theta \neq (2n+1)\frac{\pi}{2}, \quad \text{where} \quad n \in Z$

$\therefore$ Domain of secant function $= R-\{x\,|\,x=(2n+1)\frac{\pi}{2}, \quad n \in Z\}$

As sec θ attains all real values except those between -1 and 1

$\therefore$ Range of secant function $= R-\{x\,|\,-1<x<1\}$

11.1.4 Domain and Range of Cosecant Function

From figure 11.1

$\csc\theta =\neq \frac{1}{y} \quad , \quad y \quad 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OX or OX' (i.e., $X-$ axis)

$\Rightarrow \quad \theta \neq 0, \pm\pi, \pm 2\pi, ...$

$\Rightarrow \quad \theta \neq n\pi, \quad \text{where} \quad n \in Z$

$\therefore$ Domain of cosecant function $= R-\{x\,|\,x=n\pi, \quad n \in Z\}$

As csc θ attains all values except those between -1 and 1

$\therefore$ Range of cosecant function $= R-\{x\,|\,-1<x<1\}$

The following table summarizes the **domains** and **ranges** of the trigonometric functions:

Function	Domain	Range
$y=\sin x$	$-\infty<x<+\infty$	$-1\le y\le 1$
$y=\cos x$	$-\infty<x<+\infty$	$-1\le y\le 1$
$y=\tan x$	$-\infty<x<+\infty, x\neq\frac{(2n+1)\pi}{2}, n\in Z$	$-\infty<y<+\infty$
$y=\cot x$	$-\infty<x<+\infty, x\neq n\pi, \quad n\in Z$	$-\infty<y<+\infty$
$y=\sec x$	$-\infty<x<+\infty, x\neq\frac{(2n+1)\pi}{2}, n\in Z$	$y\ge 1$ or $y\le -1$
$y=\text{coses } x$	$-\infty<x<+\infty, x\neq n\pi, \quad n\in Z$	$y\ge 1$ or $y\le -1$

11.2 Period of Trigonometric Functions

All the six trigonometric functions repeat their values for each increase or decrease of 2π in θ i.e., the values of trigonometric functions for θ and $\theta \pm 2n\pi$, where $\theta\in R$, and $n\in Z$, are the same. This behaviour of trigonometric functions is called **periodicity**.

Period of a trigonometric function is the smallest $+ve$ number which, when added to the original circular measure of the angle, gives the same value of the function.

Let us now discover the periods of the trigonometric functions.

Theorem 11.1: Sine is a periodic function and its period is 2π.

Proof: Suppose p is the period of sine function such that

$\sin(\theta+p) \quad = \quad \sin\equiv\theta \quad \text{for all} \quad \theta \quad R$ (i)

Now put $\theta=0$, we have

$\sin(0+p) \quad = \quad \sin 0$

$\Rightarrow \quad \sin p = 0$

$\Rightarrow \quad p = 0, \pm\pi, \pm 2\pi, \pm 3\pi, ...$

i) if $p=\pi$, then from (i)

$\sin(\theta+\pi) \quad = \quad \sin\theta$ (not true)

$\because \quad \sin(\theta+\pi) \quad =- \quad \sin\theta$

$\therefore \quad \pi$ is not the period of sin θ.

ii) if $p=2\pi$, then from (i)

$\sin(\theta+2\pi) \quad = \quad \sin\theta,$ Which is true

As is the smallest +*ve* real number for which

$\sin(\theta + 2\pi) = \sin\theta$

$\therefore$ 2π is not the period of sin θ.

Theorem 11.2: Tangent is a periodic function and its period is π.

Proof: Suppose *p* is the period of tangent function such that

$\tan(\theta + p) = \tan\theta$ for all θ R (ii)

Now put $\theta = 0$, we have

$\tan(0 + p) = \tan 0$

$\Rightarrow \tan p = 0$

$\therefore p = 0, \pi, 2\pi, 3\pi, ...$

i) if $p = \pi$, then from (i)

$\tan(\theta + \pi) = \tan\theta$, which is true.

As π is the smallest +*ve* number for which

$\tan(\theta + \pi) = \tan\theta$

$\therefore$ π is not the period of tan θ.

Note: By adopting the procedure used in finding the periods of sine and tangent, we can prove that

i) 2π is the period of cos θ.
ii) 2π is the period of cscθ.
iii) 2π is the period of secθ.
iv) π is the period of cotθ.

Example 1: Find the periods of: i) sin $2x$ ii) $\tan\frac{x}{3}$

Solution: i) We know that the period of sine is 2π

$\therefore \sin(2x + 2\pi) = \sin 2x \quad \Rightarrow \sin 2(x + \pi) = \sin 2x$

It means that the value of sin $2x$ repeats when x is increased by π.
Hence n is the period of sin $2x$.

ii) We know that the period of tangent is π

$\therefore \tan\left(\frac{x}{3} + \pi\right) = \tan\frac{x}{3} \quad \Rightarrow \tan\frac{1}{3}(x + 3\pi) = \tan\frac{x}{3}$

It means that the value of $\tan\frac{x}{3}$ repeats when x is increased by 3π.

Hence the period of $\tan\frac{x}{3}$ is 3π.

Exercise 11.1

Find the periods of the following functions:

1.	sin $3x$	**2.**	cos $2x$	**3.**	tan $4x$	**4.**	$\cot\frac{x}{2}$
5.	$\sin\frac{x}{3}$	**6.**	$\text{coses}\frac{x}{4}$	**7.**	$\sin\frac{x}{5}$	**8.**	$\cos\frac{x}{6}$
9.	$\tan\frac{x}{7}$	**10.**	cot $8x$	**11.**	sec $9x$	**12.**	cosec $10x$.
13.	3 sin x	**14.**	2 cos x	**15.**	$3\cos\frac{x}{5}$		

11.3 Values of Trigonometric Functions

We know the values of trigonometric functions for angles of measure 0°, 30°, 45°, 60°, and 90°. We have also established the following identities:

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\sin(\pi-\theta) = \sin\theta \qquad \cos(\pi-\theta) = -\cos\theta \qquad \tan(\pi-\theta) = -\tan\theta$$
$$\sin(\pi+\theta) = -\sin\theta \qquad \cos(\pi+\theta) = -\cos\theta \qquad \tan(\pi+\theta) = \tan\theta$$
$$\sin(2\pi-\theta) = -\sin\theta \qquad \cos(2\pi-\theta) = \cos\theta \qquad \tan(2\pi-\theta) = -\tan\theta$$

By using the above identities, we can easily find the values of trigonometric functions of the angles of the following measures:

$-30^\circ, -45^\circ, -60^\circ, -90^\circ$

$\pm 120^\circ, \pm 135^\circ, \pm 150^\circ, \pm 180^\circ$

$\pm 210^\circ, \pm 225^\circ, \pm 240^\circ, \pm 270^\circ$

$\pm 300^\circ, \pm 315^\circ, \pm 330^\circ, \pm 360^\circ.$

11.4 Graphs of Trigonometric Functions

We shall now learn the method of drawing the graphs of all the six trigonometric functions. These graphs are used very often in calculus and social sciences. For graphing the linear equations of the form:

$$a_1x + b_1y + c_1 = 0 \qquad \text{(i)}$$
$$a_2x + b_2y + c_2 = 0 \qquad \text{(ii)}$$

We have been using the following procedure.

i) tables of the ordered pairs are constructed from the given equations,

ii) the points corresponding to these ordered pairs are plotted/located,

and iii) the points, representing them are joined by line segments.

Exactly the same procedure is adopted to draw the graphs of the trigonometric functions except for joining the points by the line segments.

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For this purpose,

i) table of ordered pairs (x, y) is constructed, when x is the measure of the angle and y is the value of the trigonometric ratio for the angle of measure x;

ii) The measures of the angles are taken along the X- axis;

iii) The values of the trigonometric functions are taken along the Y-axis;

iv) The points corresponding to the ordered pairs are plotted on the graph paper,

v) These points are joined with the help of **smooth ciurves**.

Note: As we shall see that the graphs of trigonometric functions will be smooth curves and none of them will be line segments or will have sharp corners or breaks **within their domains.** This behaviour of the curve is called **continuity**. It means that the trigonometric functions are continuous, wherever they are defined. Moreover, **as the trigonometric functions are periodic so their curves repeat after a fixed interval.**

11.5 Graph of y = sin x from -2π to 2π

We know that the period of sine function is 2π so, we will first draw the graph for the interval from 0° to 360° i.e., from 0 to 2π.

To graph the *sine* function, first, recall that $-1 \le \sin x \le 1$ for all $x \in R$

i.e., the range of the sine function is [-1, 1], so the graph will be between the horizontal lines y = +1 and y = – 1

The table of the ordered pairs satisfying y = sin x is as follows:

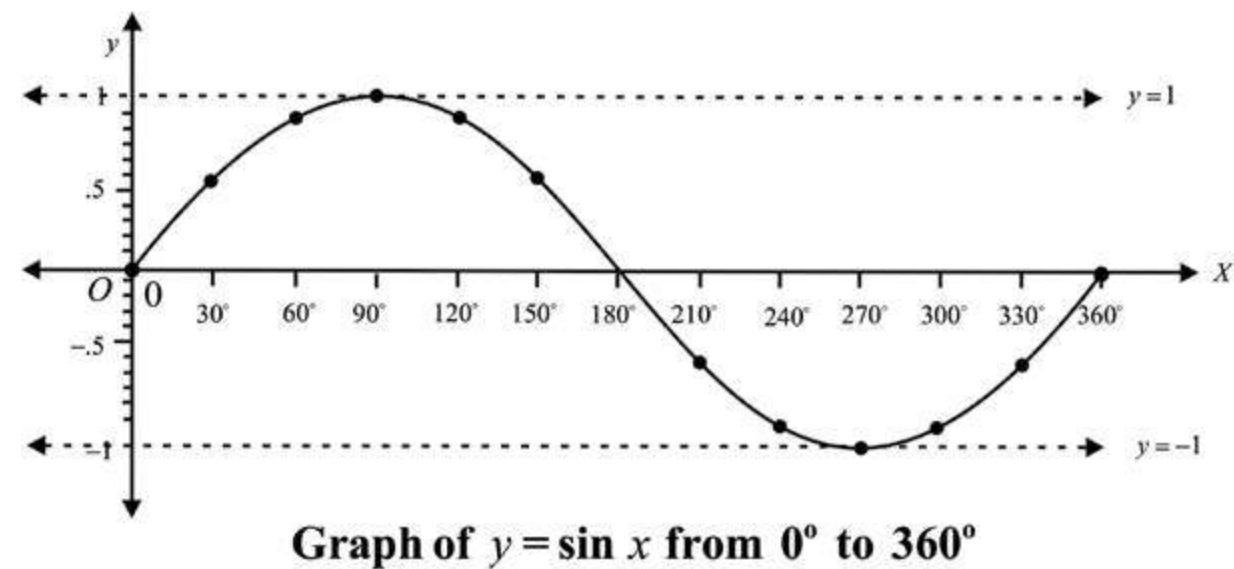
x	0 or 0°	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{3}$ or 60°	$\frac{\pi}{2}$ or 90°	$\frac{2\pi}{3}$ or 120°	$\frac{5\pi}{6}$ or 150°	π or 180°	$\frac{7\pi}{6}$ or 210°	$\frac{4\pi}{3}$ or 240°	$\frac{3\pi}{2}$ or 270°	$\frac{5\pi}{3}$ or 300°	$\frac{11\pi}{6}$ or 330°	2π or 360°
Sin x	0	0.5	0.87	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0

To draw the graph

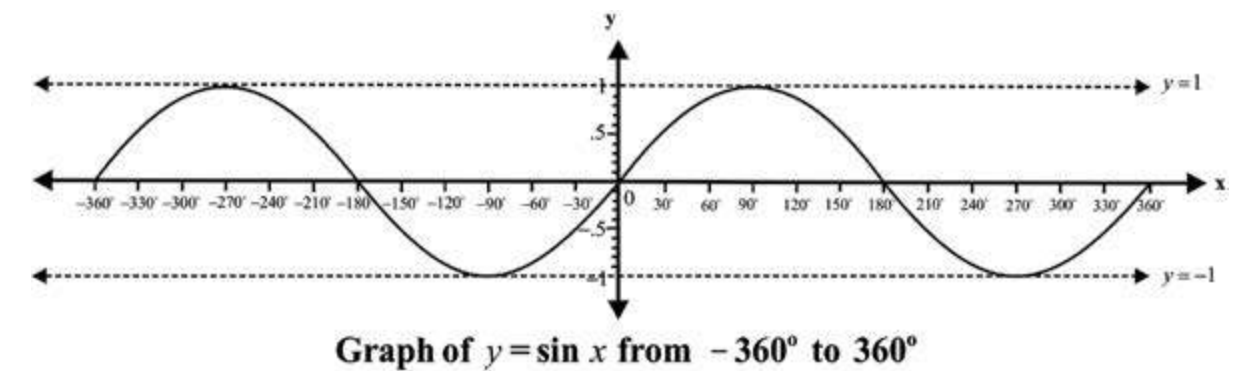
i) Take a convenient scale $\begin{cases} 1 \text{ side of small square on the } x-\text{axis} = 10^\circ \\ 1 \text{ side of big square on the } y-\text{axis} = 1 \text{ unit} \end{cases}$

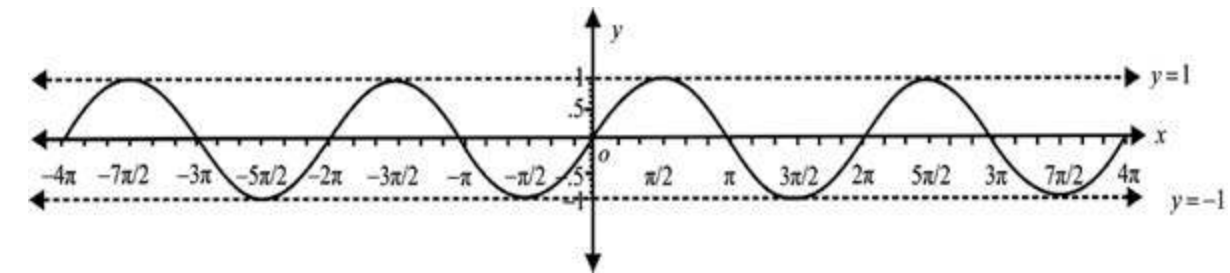
ii) Draw the coordinate axes.

iii) Plot the points corresponding to the ordered pairs in the table above i.e., (0, 0), (30°, 0.5), (60°, 0.87) and so on,

(iv) Join the points with the help of a smooth curve as shown so we get the graph of y = sin x from 0 to 360° i.e., from 0 to 2π.

Graph of $y = \sin x$ from 0° to 360°

In a similar way, we can draw the graph for the interval from 0° to –360°. This will complete the graph of y = sin x from –360° to 360° i.e. from – 2π to 2π, which is given below:

Graph of $y = \sin x$ from −360° to 360°

The graph in the interval [0, 2π] is called a cycle. Since the period of sine function is 2π, so the sine graph can be extended on both sides of x-axis through every interval of 2π (360°) as shown below:

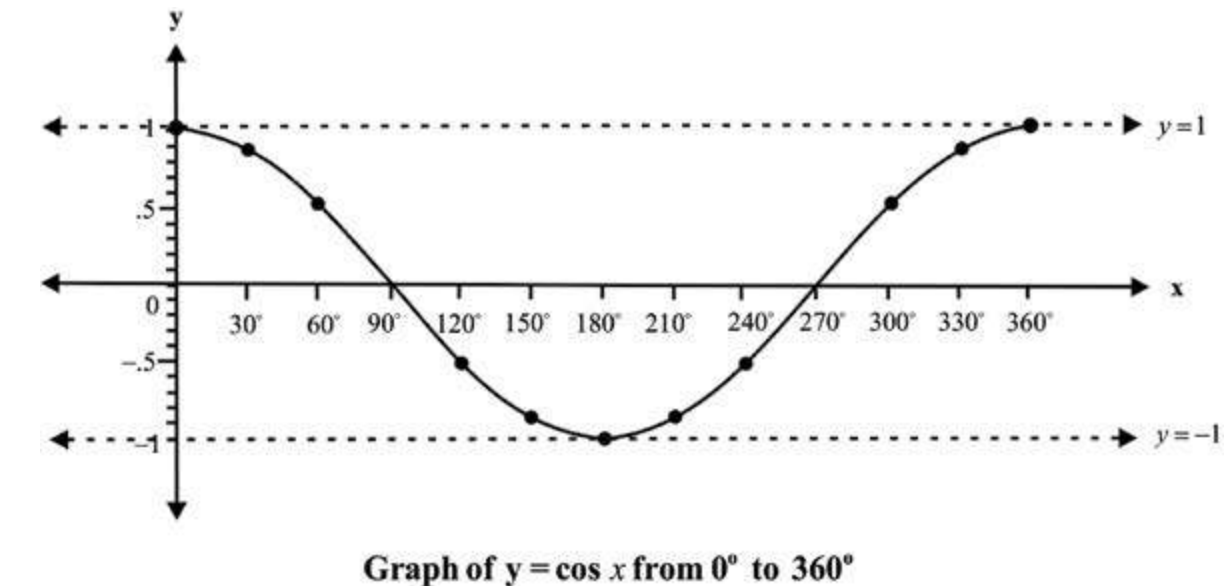
11.6 Graph of y = cos x from -2π to 2π

We know that the period of cosine function is 2π so, we will first draw the graph for the interval from 0° to 360° i.e., from 0 to 2π

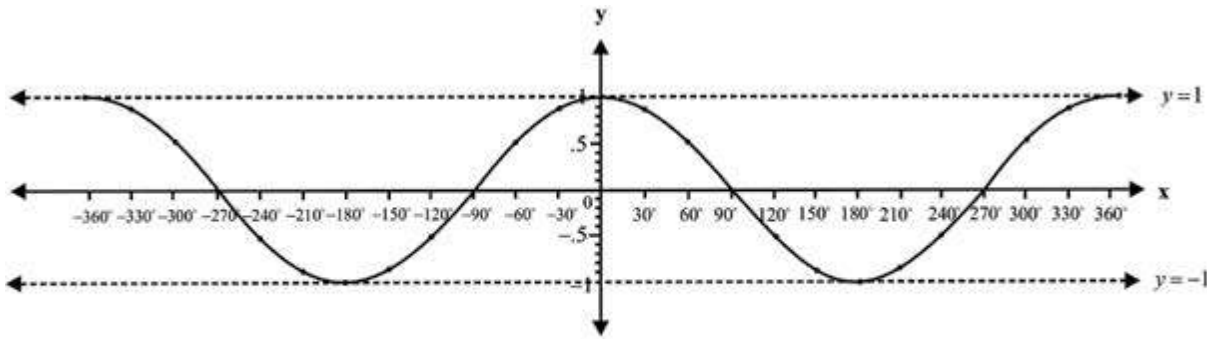
To graph the cosine function, first, recall that $-1 \le \sin x \le 1$ for all $x \in R$

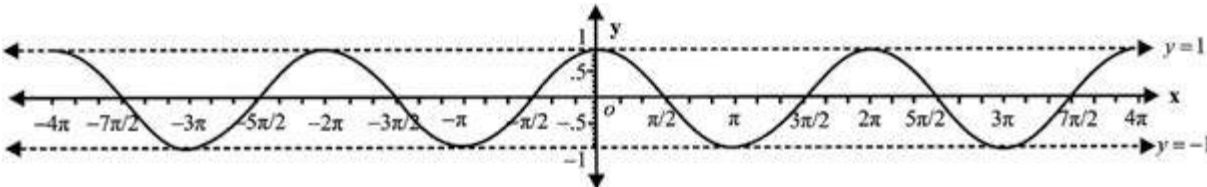
i.e., the range of the cosine function is [-1, 1], so the graph will be between the horizontal lines y = +1 and y = –1

The table of the ordered pairs satisfying y = cos x is as follows:

x	0 or 0°	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{3}$ or 60°	$\frac{\pi}{2}$ or 90°	$\frac{2\pi}{3}$ or 120°	$\frac{5\pi}{6}$ or 150°	π or 180°	$\frac{7\pi}{6}$ or 210°	$\frac{4\pi}{3}$ or 240°	$\frac{3\pi}{2}$ or 270°	$\frac{5\pi}{3}$ or 300°	$\frac{11\pi}{6}$ or 330°	2π or 360°
cos x	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1

The graph of y = cos x from 0° to 360° is given below:

Graph of y = cos x from 0° to 360°

In a similar way, we can draw the graph for the interval from 0° to –360°. This will complete the graph of y = cos x from –360° to 360° i.e. from -2π to 2π, which is given below:

Graph of $y=\cos x$ from $-360°$ to $360°$

As in the case of *sine* graph, the *cosine* graph is also extended on both sides of x-axis through an interval of 2π as shown above:

Graph of $y=\sin x$ from -4 π to 4 π

11.7 Graph of y = tan x from $-\pi$ To π

We know that tan $(-x)$ = – tan x and tan $(\pi - x)$ = – tan x, so the values of tan x for x = 0°, 30°, 45°, 60° can help us in making the table.

Also we know that tan x is undefined at x = ± 90°, when

i) x approaches $\frac{\pi}{2}$ from left i.e., $x \to \frac{\pi}{2} - 0$, tan x increases indefinitely in I Quard.

ii) x approaches $\frac{\pi}{2}$ from right i.e., $x \to \frac{\pi}{2} + 0$, tan x increases indefinitely in IV Quard.

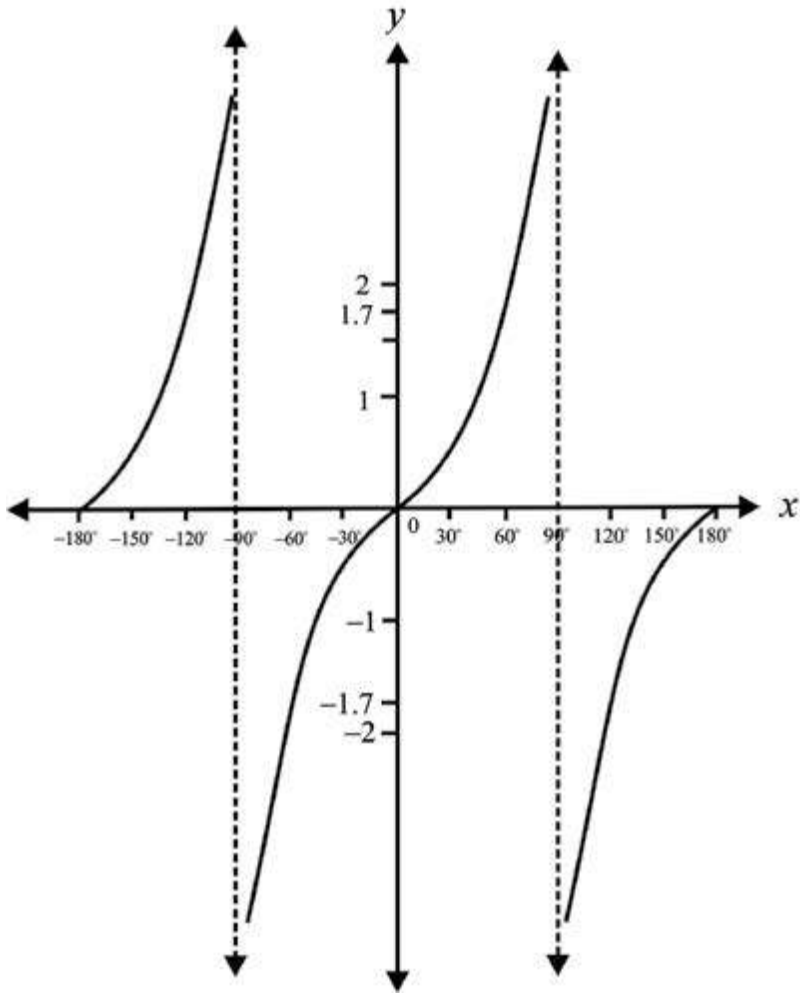
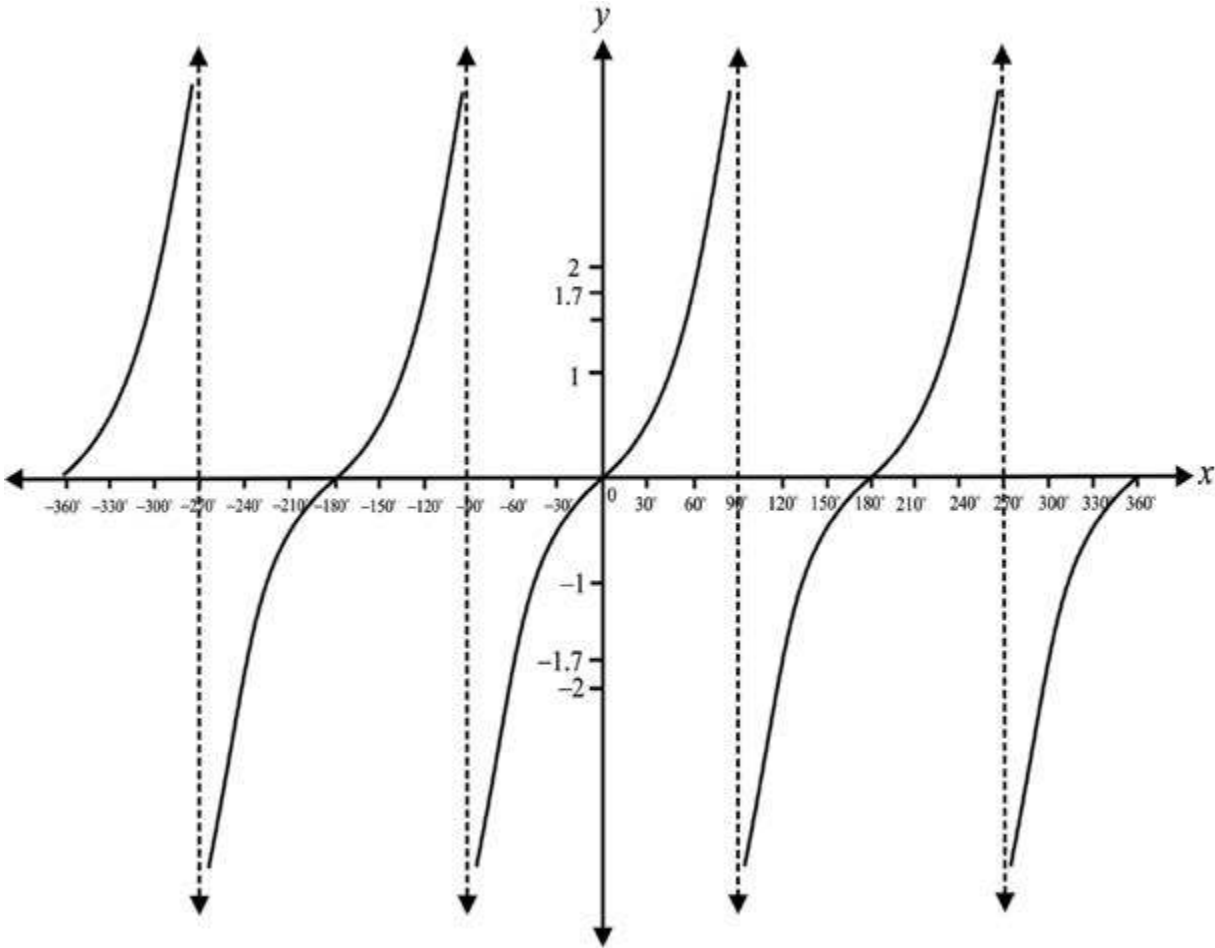
iii) x approaches $-\frac{\pi}{2}$ from left i.e., $x \to -\frac{\pi}{2} - 0$ tan x increases indefinitely in II Quard.

iv) x approaches $-\frac{\pi}{2}$ from right i.e., $x \to -\frac{\pi}{2} + 0$, tan x increases indefinitely in III Quard.

We know that the period of tangent is π , so we shall first draw the graph for the interval from $-\pi$ to π i.e., from –180° to 180°

$\therefore$ The table of ordered pairs satisfying y = tan x is given below:

x	$-\pi$ or $-180°$	$-\frac{5\pi}{6}$ or $-150°$	$-\frac{2\pi}{3}$ or $-120°$	$-\frac{\pi}{2}-0$ or $-90°-0$	$-\frac{\pi}{2}+0$ or $-90+0$	$-\frac{\pi}{3}$ or $-60°$	$-\frac{\pi}{6}$ or $-30°$	0 or 0	$\frac{\pi}{6}$ or $30°$	$\frac{\pi}{3}$ or $60°$	$\frac{\pi}{2}-0$ or $90°-0$	$\frac{\pi}{2}+0$ or $90°+0$	$\frac{2\pi}{3}$ or $120°$	$\frac{5\pi}{6}$ or $150°$	π or $180°$
Tan x	0	0.58	1.73	$+\infty$	$-\infty$	–1.73	–0.58	0	0.58	1.73	$+\infty$	$-\infty$	–1.73	–0.58	0

Graph of $y=\tan x$ from $-180°$ to $180°$

We know that the period of the tangent function is π. The graph is extended on both sides of x-axis through an interval of π in the same pattern and so we obtain the graph of y = tan x from –360° to 360° as shown below:

Graph of y = tan x from – 360° to 360°

11.8 Graph of y = cot x From -2π to π

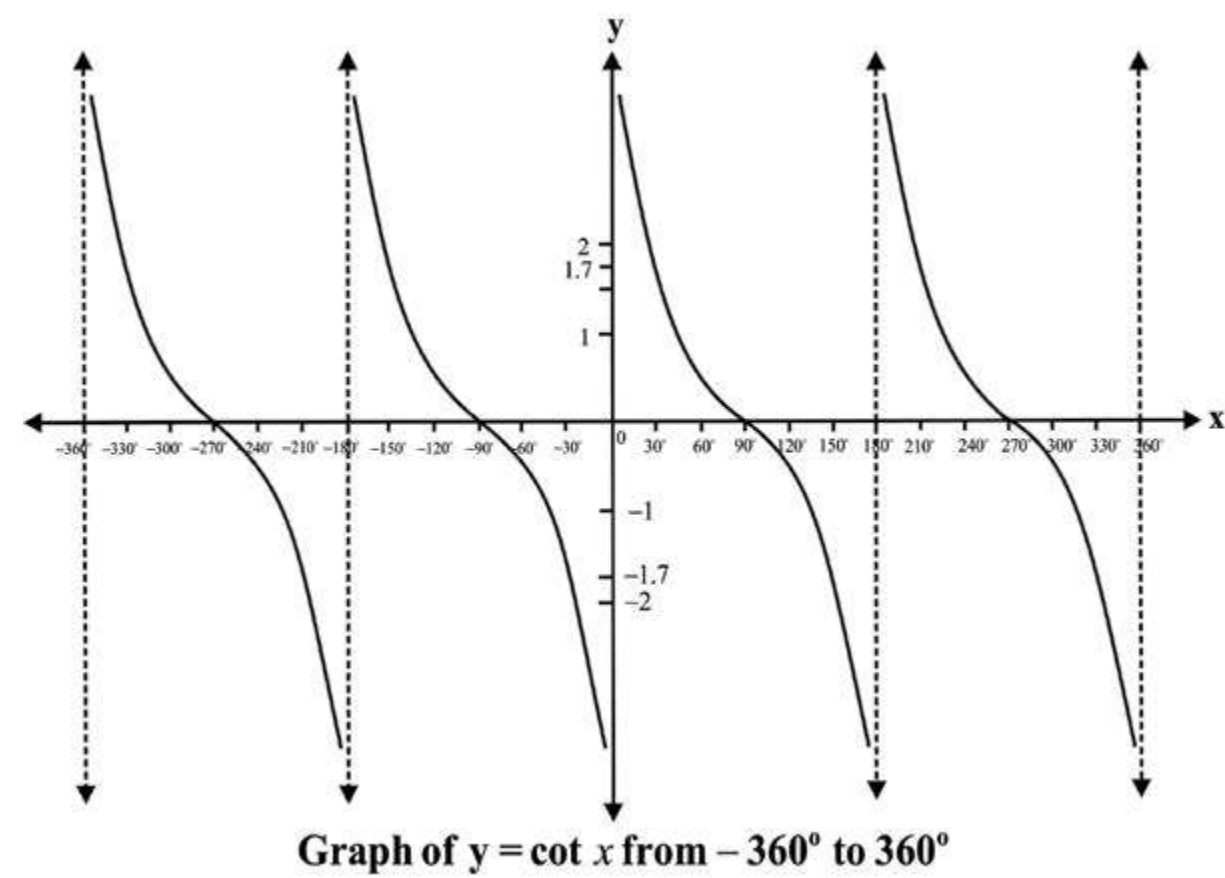
We know that cot $(-x)$ = – cot x and cot $(\pi - x)$ = – cot x, so the values of cot x for x = 0°, 30°, 45°, 60°, 90° can help us in making the table.

The period of the *cotangent function* is also π. So its graph is drawn in a similar way of *tangent* graph using the table given below for the interval from –180° to 180°.

x	$-\pi$ or $-180°$	$-\frac{5\pi}{6}$ or $-150°$	$-\frac{2\pi}{3}$ or $-120°$	$-\frac{\pi}{2}-0$ or $-90°-0$	$-\frac{\pi}{2}+0$ or $-90°+0$	$-\frac{\pi}{3}$ or $-60°$	$-\frac{\pi}{6}$ or $-30°$	0	$\frac{\pi}{6}$ or $30°$	$\frac{\pi}{3}$ or $60°$	$\frac{\pi}{2}-0$ or $90°-0$	$\frac{\pi}{2}+0$ or $90°+0$	$\frac{2\pi}{3}$ or $120°$	$\frac{5\pi}{6}$ or $150°$	π or $180°$
cot x	$\pm\infty$	1.73	0.58	$+\infty$	$-\infty$	–0.58	–1.73	$\pm\infty$	1.73	0.58	$+\infty$	$-\infty$	–0.58	–1.73	$\pm\infty$

Graph of y = cot x from – 180° to 180°

We know that the period of the cotangent function is π. The graph is extended on both

sides of x – axis through an interval of π in the same pattern and so we obtain the graph of y = cot$x$ from from –360° to 360° as shown below:

Graph of y = cot x from – 360° to 360°

11.9 Graph of y = sec x from -2π to 2π

We know that $\sec(-x) = \sec x$ and $\sec(\pi - x) = -\sec x$,

So the values of sec x for x = 0°, 30°, 45°, 60°, can help us in making the following table of the ordered pairs for drawing the graph of y = sec x for the interval 0° to 360°:

x	0 or 0	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{3}$ or 60°	$\frac{\pi}{2}-0$ or $90-0$	$\frac{\pi}{2}+0$ or $90+0$	$\frac{2\pi}{3}$ or 120°	$\frac{5\pi}{6}$ or 150°	π or 180°	$\frac{7\pi}{6}$ or 210°	$\frac{4\pi}{3}$ or 240°	$\frac{3\pi}{2}-0$ or $270-0$	$\frac{3\pi}{2}+0$ or $270+0$	$\frac{5\pi}{3}$ or 300°	$\frac{5\pi}{6}$ or 330°	2π or 360°
Sec x	1	1.15	2	∞	$-\infty$	–2	–1.15	–1	–1.15	–2	$-\infty$	$+\infty$	2	1.15	1

Since the period of sec x is also 2π, so we have the following graph of y = sec x from – 360° to 360° i.e., from -2π to 2π:

Graph of y = sec x from – 360° to 360°

11.10 Graph of y = csc x from -2π to 2π

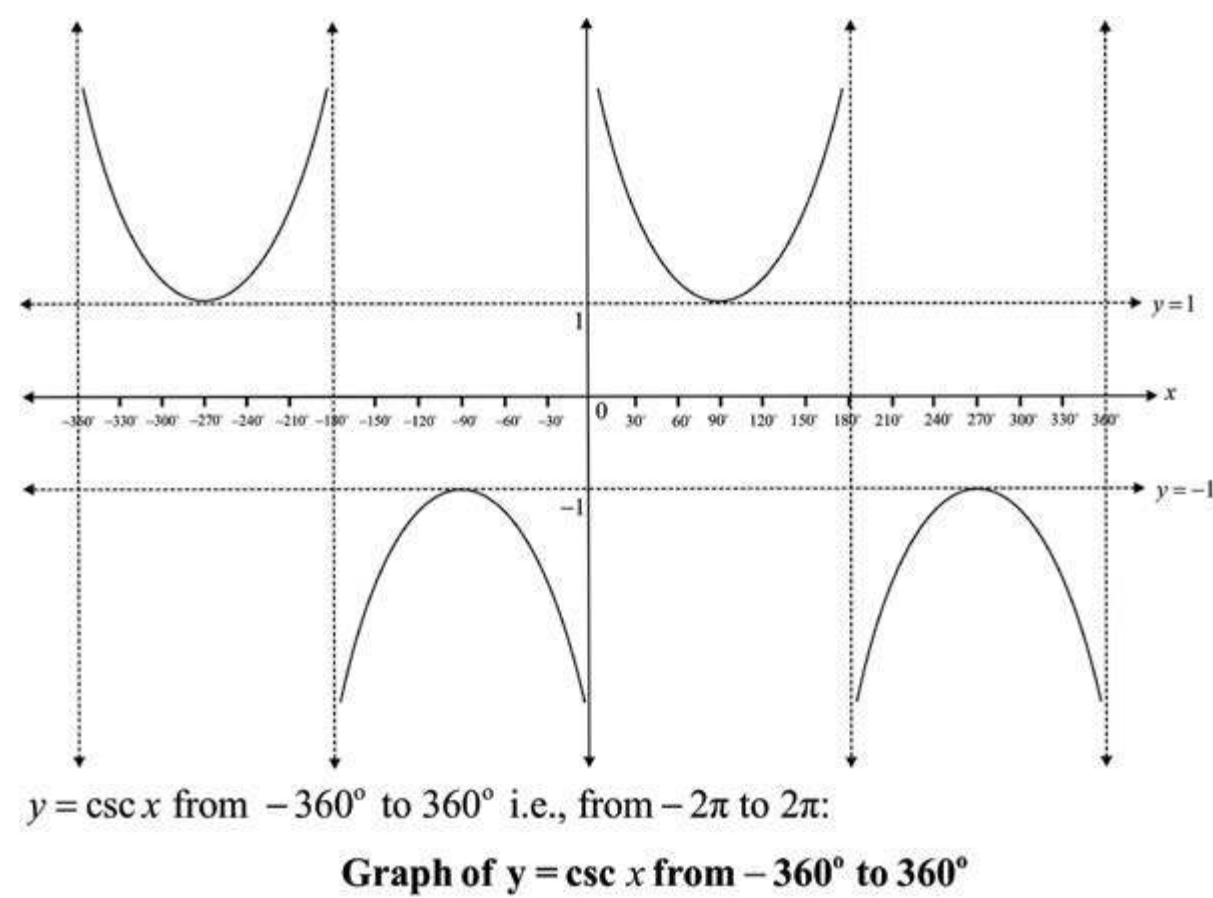
We know that: $\csc(-x) = -\csc x$ and $\csc(\pi - x) = \csc x$

So the values of csc x for x = 0°, 30°, 45°, 60°, can help us in making the following table of the ordered pairs for drawing the graph of y = csc x for the interval 0° to 360°:

x	$0+0$ or $0+0$	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{6}$ or 60°	$\frac{\pi}{2}$ or 90°	$\frac{2\pi}{3}$ or 120°	$\frac{3\pi}{6}$ or 150°	$\pi-0$ or $180-0$	$\pi+0$ or $180+0$	$\frac{7\pi}{6}$ or 210°	$\frac{4\pi}{3}$ or 240°	$\frac{3\pi}{2}$ or 270°	$\frac{5\pi}{2}$ or 300°	$\frac{4\pi}{3}$ or 330°	$2\pi-0$ or 360°
csc x	∞	2	1.15	1	1.15	2	∞	$-\infty$	–2	–1.15	–1	–1.15	–2	$-\infty$

Since the period of csc x is also 2π, so we have the following graph of

$y = \csc x$ from -360° to 360° i.e., from -2π to 2π:

Graph of y = csc x from -360° to 360°

Note 1: From the graphs of trigonometric functions we can check their *domains* and *ranges*.

Note 2: By making use of the periodic property, each one of these graphs can be extended on the left as well as on the right side of x-axis depending upon the *period* of the functions.

Note 3: The dashes lines are vertical asymptotes in the graphs of tan x, cot x, sec x and csc x.

Exercise 11.2

1. Draw the graph of each of the following function for the intervals mentioned against each :

i) $y = -\sin x, \quad x \in [-2\pi, 2\pi]$

ii) $y = \in 2\cos x, \quad x \; [0, 2\pi]$

iii) $y = \tan 2x, \quad x \; [\ \pi, \pi]$

iv) $y = \tan - x, \quad x \; [\ 2\pi, 2\pi]$

v) $y = \in \sin \frac{x}{2}, \quad x \; [0, 2\pi]$

vi) $y = \cos \frac{x}{2}, \quad x \; [\ \pi, \pi]$

2. On the same axes and to the same scale, draw the graphs of the following function for their complete period:

i) $y = \sin x$ and $y = \sin 2x$

ii) $y = \cos x$ and $y = \cos 2x$

3. Solve graphically:

i) $\sin x = \cos x, \quad x \; [0, \pi]$

ii) $\sin x = \in x \ , \quad x \; [0, \pi]$

CHAPTER

12

Application of Trigonometry

12.1 Introduction

A triangle has six important elements; three angles and three sides. In a triangle *ABC*, the measures of the three angles are usually denoted by α, β, γ and the measures of the three sides opposite to them are denoted by *a, b, c* respectively.

If any three out of these six elements, out of which atleast one side, are given, the remaining three elements can be determined This process of finding the unknown elements is called the **solution of the triangle.**

We have calculated the values of the trigonometric functions of the angles measuring 0°, 30°, 45°, 60° and 90°. But in a triangle, the angles are not necessarily of these few measures. So, in the solution of triangles, we may have to solve problems involving angles of measures other than these. In such cases, we shall have to consult natural sin/cos/tan tables or we may use $\boxed{\sin}, \boxed{\cos}, \boxed{\tan}$ keys on the calculator.

Tables/calculator will also be used for finding the measures of the angles when value of trigonometric ratios are given e.g. to find θ when $\sin\theta = x$.

12.2 Tables of Trigonometric Ratios

Mathematicians have constructed tables giving the values of the trigonometric ratios of large number of angles between 0° and 90°. These are called tables of natural **sines, cosines, tangents** etc. In four-figure tables, the interval is 6 minutes and difference corresponding to 1,2, 3, 4, 5 minutes are given in the **difference columns.**
The following examples will illustrate how to consult these tables.

Example 1: Find the value of
i) sin 38° 24′ ii) sin 38° 28′ iii) tan 65° 30′.

Solution: In the first column on the left hand side headed by degrees (in the Natural Sine table) we read the number 38°. Looking along the row of 38° till the minute column number 24′ is reached, we get the number 0.6211.

$\therefore \quad \sin 38^\circ 24' = 0.6211$

ii) To find sin 38° 28′, we first find sin 38° 24′, and then see the right hand column headed by **mean differences.** Running down the column under 4′ till the row of 38° is reached. We find 9 as the difference for 4′. Adding 9 to 6211, we get 6220.

$\therefore \quad \sin 38^\circ 24' = 0.6220$

Note: 1. As $\sin\theta$, $\sec\theta$ and $\tan\theta$ go on increasing as θ increases from 0° to 90°, so the numbers in the columns of the differences for $\sin\theta$, $\sec\theta$ and $\tan\theta$ are added.
2. Since $\cos\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$ decrease as θ increases from 0° to 90°, therefore, for $\cos\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$ the numbers in the column of the, differences are subtracted.

iii) Turning to the tables of Natural Tangents read the number 65° in the first column on the left hand side headed by degrees. Looking along the row of 65° till the minute column under 30′ is reached, we get the number 1943. The integral part of the figure just next to 65° in the horizontal line is 2.

$\therefore \quad \tan 65^\circ 30' = 2.1943$

Example 2: If $\sin x = 0.5100$, find x.

Solution: In the tables of Natural Sines, we get the number (nearest to 5100) 5090 which lies at the intersection of the row beginning with 30° and the column headed by 36′. The difference between 5100 and 5090 is 10 which occurs in the row of 30° under the mean difference column headed by 4′. So, we add 4′ to 30° 36′ and get

$\sin^{-1}(0.5100) = 30^\circ 40'$

Hence x = 30° 40

Exercise 12.1

1. Find the values of:

i)	$\sin 53^\circ 40'$	ii)	$\cos 36^\circ 20'$	iii)	$\tan 19^\circ 30'$
iv)	$\cot 33^\circ 50'$	v)	$\cos 42^\circ 38'$	vi)	$\tan 25^\circ 34'$
vii)	$\sin 18^\circ 31'$	viii)	$\cos 52^\circ 13'$	ix)	$\cot 89^\circ 9'$

2. Find θ, if:

i) $\sin\theta = 0.5791$		ii) $\cos\theta = 0.9316$
iii) $\cos\theta = 0.5257$		iv) $\tan\theta = 1.705$
v) $\tan\theta = 21.943$		vi) $\sin\theta = 0.5186$

12.3 Solution of Right Triangles

In order to solve a right triangle, we have to find:

i) the measures of two acute angles

and ii) the lengths of the three sides.

We know that a trigonometric ratio of an acute angle of a right triangle involves 3 quantities "lengths of two sides and measure of an angle". Thus if two out of these three quantities are known, we can find the third quantity.

Let us consider the following two cases in solving a right triangle:

CASE I: When Measures of Two Sides are Given

Example 1: Solve the right triangle ABC, in which b = 30.8, c = 37.2 and γ= 90°**.**

Solution: From the figure,

$$\cos\alpha = \frac{b}{c} = \frac{30.8}{37.2} = 0.8280$$

$\Rightarrow \quad \alpha = \cos^{-1} 0.8280 = 34^\circ 6'$

$\because \quad \gamma = 90^\circ \quad \Rightarrow \beta = 90^\circ - \alpha = 90^\circ - 34^\circ 6, = 55^\circ 54,$

$\because \quad \frac{a}{c} = \sin\alpha$

$\Rightarrow \quad a = c\sin\alpha \quad 37.2\sin 34^\circ 6$

$= 37.2(0.5606)$

$= 20.855$

$\Rightarrow \quad a = 20.9$

Hence $\quad a = 20.9, \quad \alpha = 34^\circ$ and $\beta = 55^\circ 54$

CASE II: When Measures of One Side and One Angle are Given

Example 2: Solve the right triangle, in which

$\alpha = 58^\circ 13'$, $b = 125.7$ and $\gamma = 90^\circ$

Solution: $\because \gamma = 90^\circ, \alpha = 58^\circ 13' \quad \therefore \quad \beta = 90^\circ - 58^\circ 13' = 31^\circ 47'$

From the figure,

$\frac{a}{b} = \tan 58^\circ \ 13'$

$\Rightarrow \quad a = (125.7)\tan 58^\circ 13'$

$= 125.7(1.6139)$

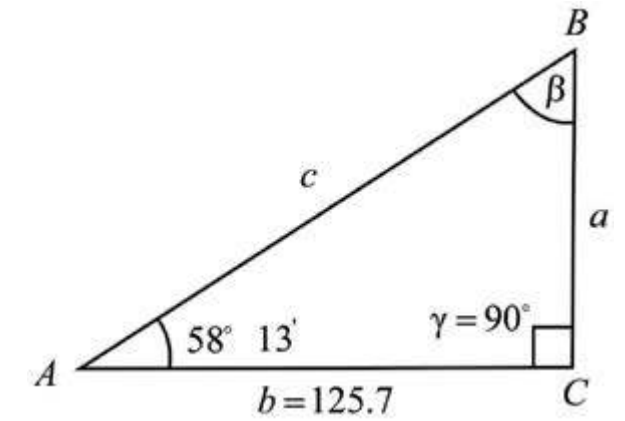
$= 202.865$

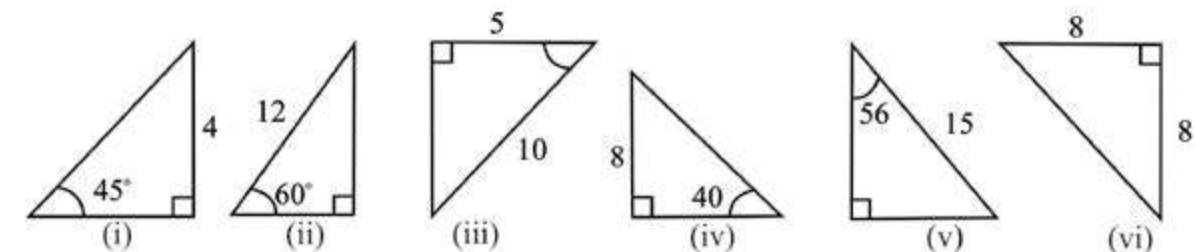
$\therefore \quad a = 202.9$

Again $\quad \frac{a}{c} = \sin 58^\circ 13'$

$\Rightarrow \quad c = \frac{202.9}{0.8500}$

$\therefore \quad c = 238.7$

Hence $\quad a = 202.9, \ \beta \quad 31^\circ 47'$ and $c \quad 238.7$

Exercise 12.2

1. Find the unknown angles and sides of the following triangles:

Solve the right triangle *ABC*, in which $\gamma = 90^\circ$

2. $\alpha = 37^\circ 20'$, $a \quad 243$ **3.** $\alpha = 62^\circ 40'$, $b \quad 796$

4. *a* = 3.28, *b* =5.74 **5.** *b* = 68.4, *c* = 96.2

6. a = 5429, c = 6294 **7.** $\beta = 50^{\circ}10'$, c 0.832

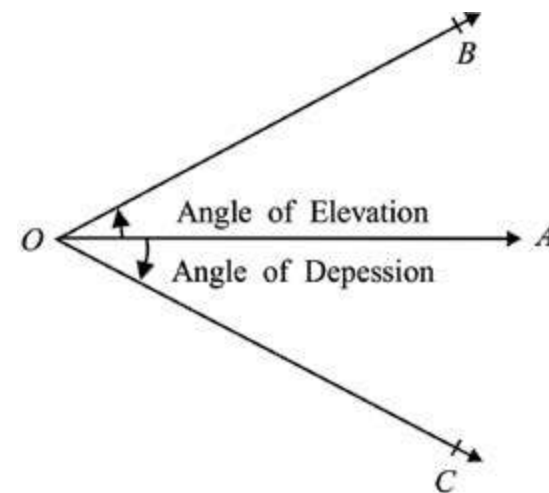
12.4 (a) Heights And Distances

One of the chief advantages of trigonometry lies in finding heights and distances of inaccessible objecst:

In order to solve such problems, the following procedure is adopted:

1) Construct a clear labelled diagram, showing the known measurements.
2) Establish the relationships between the quantities in the diagram to form equations containing trigonometric ratios.
3) Use tables or calculator to find the solution.

(b) Angles of Elevation and Depression

If $\overrightarrow{OA}$ is the horizontal ray through the eye of the observer at point O, and there are two objects B and C such that B is above and C is below the horizontal ray $\overrightarrow{OA}$, then,

i) for looking at B above the horizontal ray, we have to raise our eye , and $\angle AOB$ is called the **Angle of Elevation** and

ii) for looking at C below the horizontal ray we have to lower our eye , and $\angle AOC$ is called the **Angle of Depression.**

Example 1: A string of a flying kite is 200 meters long, and its angle of elevation is 60°. Find the height of the kite above the ground taking the string to be fully stretched.

Solution: Let O be the position of the observer, B be the position of the kite and $\overrightarrow{OA}$ be the horizontal ray through O.

Draw $\overline{BA} \perp \overrightarrow{OA}$

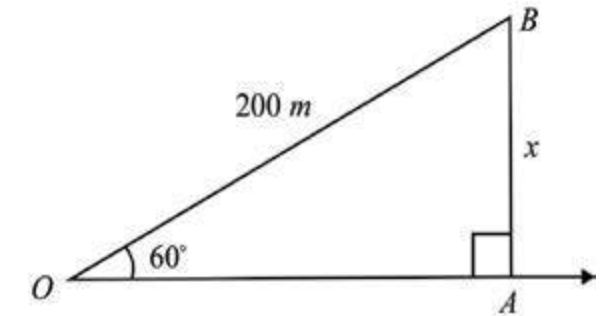
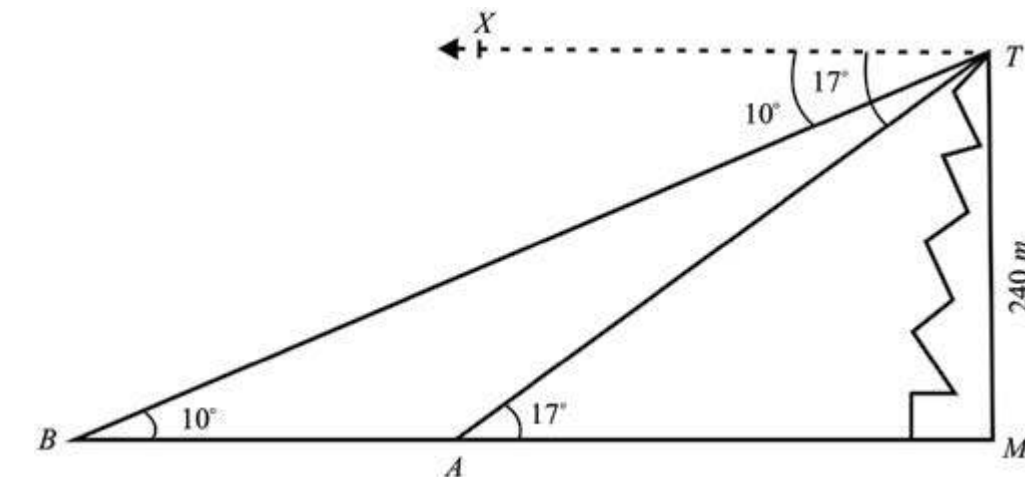
Now $m\angle O = 60^{\circ}$ and $OB = 200m$

Suppose $AB = x$ meters

In right ΔOAB,

$$\frac{x}{200} = \sin 60^{\circ} = \frac{\sqrt{3}}{2} = \frac{1.732}{2}$$

$$\Rightarrow \quad x = 200\left(\frac{1.732}{2}\right) = 100(1.732) = 173.2$$

Hence the height of the kite above the ground = 173.2 m.

Example 2: A surveyor stands on the top of 240 m high hill by the side of a lake. He observes two boats at the angles of depression of measures 17° and 10°. If the boats are in the same straight line with the foot of the hill just below the observer, find the distance between the two boats, if they are on the same side of the hill.

Solution: Let T be the top of the hill $\overline{TM}$, where the observer is stationed, A and B be the positions of the two boats so that $m\angle XTB$ = 10° and $m\angle XTA$ = 17° and TM = 240m :

Now, $m\angle MAT = \angle m\ XTA = 17^{\circ}$ $(\because \overline{TX} \,|\, |\overline{MA})$

and $m\angle MBT = m\angle XTB = 10^{\circ}$ $(\because \overline{TX} \,|\, |\overline{MA})$

From the figure, $\dfrac{\overline{TM}}{\overline{AM}} = \tan 17^{\circ}$

$$\Rightarrow \overline{AM} = \frac{\overline{TM}}{\tan 17^\circ} = \frac{240}{0.3057}$$

$$\Rightarrow \overline{AM} = 785m$$

and $\frac{\overline{TM}}{\overline{BM}} = \tan 10^\circ$

$$\Rightarrow \overline{BM} = \frac{\overline{TM}}{\tan 10^\circ} = \frac{240}{0.1763} = 1361m$$

$$\therefore \overline{AB} = \overline{BM} - \overline{AM} = 1361 - 785 = 576m$$

Hence the distance between the boats = 576m.

Example 3: From a point 100 m above the surface of a lake, the angle of elevation of a peak of a cliff is found to be 15° and the angle of depression of the image of the peak is 30°. Find the height of the peak.

Solution:

Let A be the top of,the peak $\overline{AM}$ and $\overline{MB}$ be its image. Let P be the point of observation and L be the point just below P (on the surface of the lake).

such that $\overline{PL}$ = 100m

From P, draw $\overline{PQ} \perp \overline{AM}$.

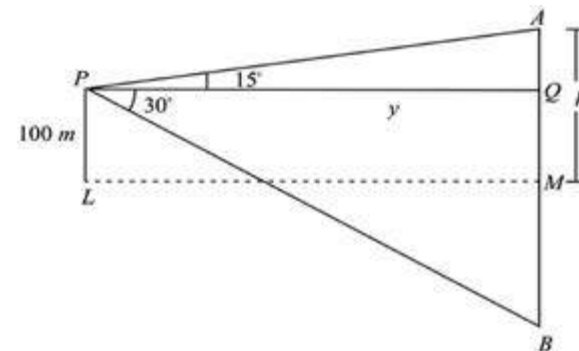
Let $\overline{PQ} = y$ metres and $\overline{AM} = h$ metres.

$$\therefore \overline{AQ} = h - \overline{QM} = h - \overline{PL} = h - 100$$

From the figure,

$$\tan 15^\circ = \frac{\overline{AQ}}{\overline{PQ}} = \frac{h-100}{y} \quad \text{and} \quad \tan 30^\circ = \frac{\overline{BQ}}{\overline{PQ}} = \frac{100+h}{y}$$

By division, we get

$$\frac{\tan 15^\circ}{\tan 30^\circ} = \frac{h-100}{h+100}$$

By Componendo and Dividendo, we have

$$\frac{\tan 15^\circ + \tan 30^\circ}{\tan 15^\circ - \tan 30} = \frac{h-100+h+100}{h-100-h-100} = \frac{2h}{-200} = \frac{h}{-100}$$

$$\therefore h = \frac{\tan 30^\circ + \tan 15^\circ}{\tan 30^\circ - \tan 15} \times 100 = \left[\frac{0.5774+0.2679}{0.5774-0.2679}\right] 100$$

$$\Rightarrow h = 273.1179.$$

Hence height of the peak = 273 m. (Approximately)

12.5 Engineering and Heights and Distances

Engineers have to design the construction of roads and tunnels for which the knowledge of heights and distance is very useful to them. Moreover, they are also required to find the heights and distances of the out of reach objects.

Example 4: An O.P., sitting on a cliff 1900 meters high, finds himself in the same vertical plane with an anti-air-craft gun and an ammunition depot of the enemy. He observes that the angles of depression of the gun and the depot are 60° and 30° respectively. He passes this information on to the headquarters. Calculate the distance between the gun and the depot.

Solution: Let O be the position of the O.P., A be the point on the ground just below him and B and C be the positions of the gun and the depot respectively.

$$\overline{OA} = 1900m$$

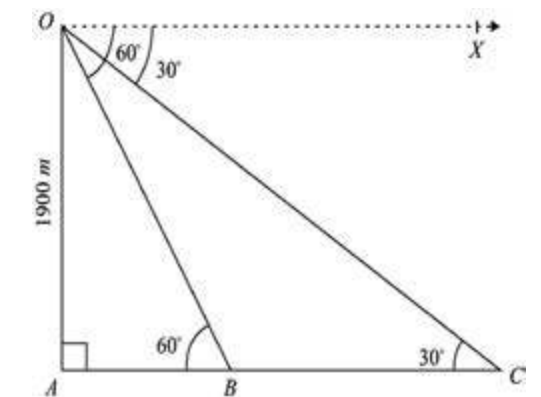
$$m\angle BOX = 60^\circ$$

and $m\angle COX = 30^\circ$

$$\Rightarrow m\angle ABO = m\angle BOX = 60^\circ, \quad m\angle ACO = 30^\circ$$

In right ΔBAO, In right ΔCAO,

$$\frac{1900}{\overline{AB}} = \tan 60^\circ \qquad \frac{1900}{\overline{AC}} = \tan 30^\circ$$

$$\Rightarrow \overline{AB} = \frac{1900}{\tan 60^\circ} = \frac{1900}{\sqrt{3}} \qquad \overline{AC} = \frac{1900}{\tan 30^\circ}$$

Now $\overline{BC} = \overline{AC} - \overline{AB}$ $\Rightarrow$ $\overline{AC} = 1900\sqrt{3}$

$$\Rightarrow \overline{BC} = 1900\sqrt{3} - \frac{1900}{\sqrt{3}} = 2193.93$$

$\therefore$ Required distance = 2194 meters.

Exercise 12.3

1. A vertical pole is 8 m high and the length of its shadow is 6 m. What is the angle of elevation of the sun at that moment?
2. A man 18 dm tall observes that the angle of elevation of the top of a tree at a distance of 12 m from him is 32 . What is the height of the tree?
3. At the top of a cliff 80 m high, the angle of depression of a boat is 12°. How far is the boat from the cliff?
4. A ladder leaning against a vertical wall makes an angle of 24° with the wall. Its foot is 5m from the wall. Find its length.
5. A kite flying at a height of 67.2 m is attached to a fully stretched string inclined at an angle of 55° to the horizontal. Find the length of the string.
6. When the angle between the ground and the suri is 30°, flag pole casts a shadow of 40m long. Find the height of the top of the flag.
7. A plane flying directly above a post 6000 m away from an anti-aircraft gun observes the gun at an angle of depression of 27°.Find the height of the plane.
8. A man on the top of a 100 m high light-house is in line with two ships on the same side of it, whose angles of depression from the man are 17° and 19° respecting. Find the distance between the ships.
9. *P* and *Q* are two points in line with a tree. If the distance between *P* and *Q* be 30 m and the angles of elevation of the top of the tree at *P* and *Q* be 12° and l5° respectively, find the height of the tree.
10 Two men are on the opposite sides of a 100 m high tower. If the measures of the angles of elevation of the top of the tower are 18° and 22°respectively find the distance between them.
11. A man standing 60 m away from a tower notices that the angles of elevation of the top and the bottom of a flag staff on the top of the tower are 64° and 62° respectively. Find

the length of the flag staff.

12. The angle of elevation of the top of a 60 m high tower from a point *A*, on the same level as the foot of the tower, is 25°. Find the angle of elevation of the top of the tower from a point *B*, 20 m nearer to *A* from the foot of the tower.
13. Two buildings *A* and *B* are 100 m apart. The angle of elevation from the top of the building *A* to the top of the building *B* is 20°. The angle of elevation from the base of the building *B* to the top of the building *A* is 50°. Find the height of the building *B*.
14. A window washer is working in a hotel building. An observer at a distance of 20 m from the building finds the angle of elevation of the worker to be of 30°. The worker climbs up 12 m and the observer moves 4 m farther away from the building. Find the new angle of elevatiqn of the worker.
15 A man standing on the bank of a canal observes that the measure of the angle of elevation of a tree on the other side of the canal, is 60. On retreating 40 meters from the bank, he finds the measure of the angle of elevation of the tree as 30 . Find the height of the tree and the width of the canal.

12.6 Oblique Triangles

A triangle, which is not right, is called an oblique triangle. Following triangles are not right, and so each one of them is oblique:

We have learnt the methods of solving right triangles. However, in solving oblique triangles, we have to make use of the relations between the sides *a, b, c* and the angle α,β,γ of such triangles, which are called **law of cosine, law of sines** and **law of tangents**.

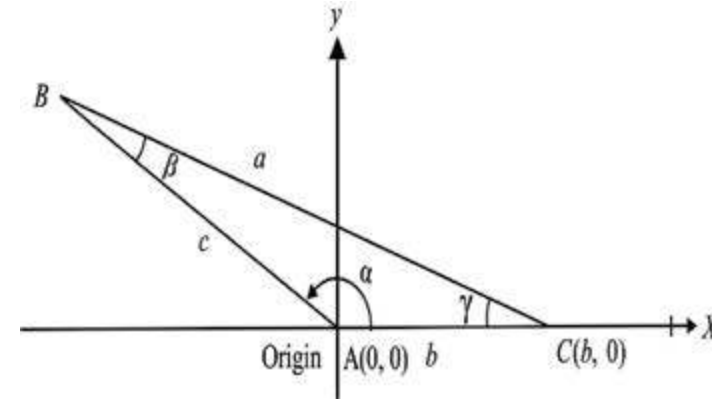
Let us discover these laws one by one before solving oblique triangles.

12.6.1 The Law of Cosine

In any triangle *ABC*, with usual notations, prove that:

i) $a^2 = b^2 + c^2 - 2bc\cos\alpha$

ii) $b^2 = c^2 + a^2 - 2ca\cos\beta$

iii) $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Proof: Let side $\overline{AC}$ of triangle *ABC* be along the positive direction of the x-axis with vertex *A* at origin, then $\angle BAC$ will be in the standard position.

$\because \quad \overline{AB} = c$ and $m\angle BAC = \alpha$

$\therefore$ coodinates of B are $(c\cos\alpha, c\sin\alpha)$

$\because$ *AC* = *b* and point *C* is on the x-axis

$\therefore$ Coordinates of *C* are (*b*, 0)

By distance formula,

$$|\overline{BC}|^2 = (c\cos\alpha - b)^2 + (c\sin\alpha - 0)^2$$

$$\Rightarrow \quad a^2 = c^2\cos^2\alpha + b^2 - 2bc\cos\alpha + c^2\sin^2\alpha \quad \left(\because \overline{BC} \quad a\right)$$

$$\Rightarrow \quad a^2 = c^2(\cos^2\alpha + \sin^2\alpha) + b^2 - 2bc\cos\alpha$$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos\alpha \qquad \text{(i)}$$

In a similar way, we can prove that

$$b^2 = c^2 + a^2 - 2ca\cos\beta \qquad \text{(ii)}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \qquad \text{(iii)}$$

(i), (ii) and (iii) are called law of cosine. They can also be expressed as:

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\beta = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

Note: If ΔABC is right, then

Law of cosine reduces to Pythagorous Theorem i.e.,

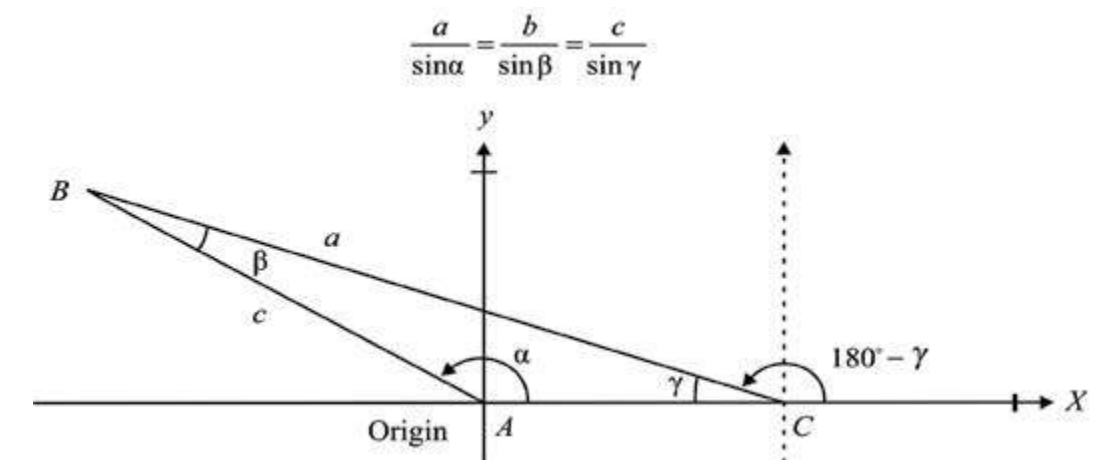
if $\alpha = 90^\circ$ = then $b^2 \quad c^2 \quad a^2$

or if $\beta = 90^\circ$ = then $c^2 \quad a^2 \quad b^2$

or if $\gamma = 90^\circ$ = then $a^2 \quad b^2 \quad c^2$

12.6.2 The Law of Sines

In any triangle *ABC*, with usual notations, prove that:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Proof: Let side $\overline{AC}$ of.triangle *ABC* be along the positive direction of the x-axis with vertex *A* at origin, then $\angle BAC$ will be in the standard position.

$\therefore \quad \overline{AB} = c \quad \text{and} \quad m\angle BAC = \alpha$

$\therefore$ The coodinates of the point B are $(c\cos\alpha, c\sin\alpha)$

If the origin A is shifted to C, then $\angle BCX$ will be in the standard position,

$\because \quad \overline{BC} = a \quad \text{and} \quad m\angle BCX = 180^\circ - \gamma$

$\therefore$ The coodinates of B are $[a\cos(180^\circ - \gamma), a\sin(180^\circ - \gamma)]$

In both the cases, the y-coordinate of B remains the same

$\Rightarrow \quad a\sin(180 - \gamma) = c\sin\alpha$

$a\sin\gamma = c\sin\alpha$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \qquad \text{(i)}$$

In a similar way, with side $\overline{AB}$ along +ve x-axis, we can prove that:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \qquad \text{(ii)}$$

From (i) and (ii), we have $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

This is called the law of sines.

12.6.3 The Law of Tangents

In any triangle ABC, with usual notations, prove that:

i) $\dfrac{a-b}{a+b} = \dfrac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}}$ ii) $\dfrac{b-c}{b+c} = \dfrac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}}$

iii) $\dfrac{c-a}{c+a} = \dfrac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}}$

Proof: We know that by the law of sines:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\Rightarrow \frac{a}{b} = \frac{\sin\alpha}{\sin\beta}$$

By componendo and dividendo,

$$\frac{a-b}{a+b} = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}} \qquad \text{(i)}$$

Similarly, we can prove that:

$$\frac{b-c}{b+c} = \frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}} \quad \text{(ii)} \qquad \text{and} \qquad \frac{c-a}{c+a} = \frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}} \quad \text{(iii)}$$

$\Rightarrow$ (i), (ii) and (iii) are called **Law of Tangents.**

12.6.4 Half Angle Formulas

We shall now prove some more formulas with the help of the law of cosine, which are called half-angle formulas:

a) The Sine of Half the Angle in Terms of the Sides

In any triangle *ABC*, prove that :

$$\left.\begin{array}{ll}\text{(i)} & \sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \\ \text{(ii)} & \sin\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}} \\ \text{(iii)} & \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}\end{array}\right\} \quad \text{where } 2s = a+b+c$$

Proof: We know that

$$2\sin^2\frac{\alpha}{2} = 1-\cos\alpha$$

$$\therefore \quad 2\sin^2\frac{\alpha}{2} = 1 \quad \frac{b^2+c^2-a^2}{2bc} \qquad \left\{\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc}\right.$$

$$= \frac{2bc-b^2-c^2+a^2}{2bc}$$

$$\therefore \quad 2\sin^2\frac{\alpha}{2} = \frac{a^2-(b^2+c^2-2bc)}{2bc} = \frac{a^2-(b-a)^2}{2bc}$$

$$\therefore \quad \sin^2\frac{\alpha}{2} = \frac{(a+b-c)(a-b+c)}{4bc}$$

$$\therefore \quad \sin^2\frac{\alpha}{2} = \frac{2(s-c).2(s-b)}{4bc} \qquad \{\because a \quad b \quad c \quad 2s\}$$

Hence: $\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ $\qquad \left\{\begin{array}{l}\text{is the measure of} \\ \text{an angle of } ABC \\ \therefore \frac{\alpha}{2} < 90 \Rightarrow \sin\frac{\alpha}{2} = \text{ ve}\end{array}\right.$

In a similar way, we can prove that

$$\sin\frac{\beta}{2} == \sqrt{\frac{(s-c)(s-a)}{ca}} \quad \text{and} \quad \sin\frac{\gamma}{2} \quad \sqrt{\frac{(s-a)(s-b)}{ab}}$$

b) The Cosine of Half the Angle in Term of the Sides

In any triangle *ABC*, with usual notation, prove that:

$$\left.\begin{array}{ll}\text{i)} & \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}} \\ \text{ii)} & \cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}} \\ \text{iii)} & \cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}\end{array}\right\} \quad \text{where } 2s = a+b+c$$

Proof: We know that

$$2\cos^2\frac{\alpha}{2} = 1+\cos\alpha = 1 \quad \frac{b^2+c^2-a^2}{2bc} \qquad \left[\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc}\right]$$

$$== \frac{2bc+b^2+c^2-a^2}{2bc} \qquad \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc}$$

$$\therefore \quad \cos^2\frac{\alpha}{2} = \frac{(a+b+c)(b+c-a)}{4bc}$$

$$\therefore \quad \cos^2\frac{\alpha}{2} = \frac{2s.2(s-a)}{4bc} \qquad (\therefore 2s = a+b+c)$$

$$\Rightarrow \quad \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}} \qquad \left\{\begin{array}{l}\because \alpha \text{ is measure of} \\ \text{an angle of } ABC \\ \therefore \frac{\alpha}{2} \text{ is acute } \Rightarrow \cos\frac{\alpha}{2} = \text{ ve}\end{array}\right.$$

In a similar way, we can prove that

$$\cos\frac{\beta}{2} == \sqrt{\frac{s(s-b)}{ca}} \quad \text{and} \quad \cos\frac{\gamma}{2} \quad \sqrt{\frac{s(s-c)}{ab}}$$

c) The Tangent of Half the Angle in Terms of the Sides

In any triangle *ABC*, with usual notation, prove that:

$$\left.\begin{array}{ll}\text{(i)} & \tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\ \text{(ii)} & \tan\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \\ \text{(iii)} & \tan\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\end{array}\right\} \quad \text{where } 2s = a+b+c$$

Proof: We know that:

$$\sin\frac{\alpha}{2} == \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \text{and} \quad \cos\frac{\alpha}{2} \quad \sqrt{\frac{s(s-a)}{bc}}$$

$$\Rightarrow \quad \tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\beta}{2}} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$\therefore \quad \tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

In a similar way, we can prove that:

$$\tan\frac{\beta}{2} == \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \quad \text{and} \quad \tan\frac{\gamma}{2} \quad \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

12.7 Solution of Oblique Triangles

We know that a triangle can be constructed if:

i) one side and two angles are given,

or ii) two sides and their included angle are given

or iii) three sides are given.

In the same way, we can solve an oblique triangle if

i) one side and two angles are known,

or ii) two sides and their included angle are known

or iii) three sides are known.

Now we shall discover the methods of solving an oblique triangle in each of the above cases:

12.7.1 Case I: When measures of one side and two angles are given

In this case, the law of sines can be applied.

Example 1: Solve the triangle *ABC*, given that

$$\alpha = 35^\circ 17', \quad \beta = 45^\circ 13', \quad b = 421$$

Solution: $\because \alpha + \beta + \gamma = 180^\circ$

$$\therefore \quad \gamma = 180^\circ - (\alpha + \beta) = 180^\circ - (35^\circ 17' + 45^\circ 13') = 99^\circ 30'$$

By Law of sines, we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\Rightarrow \quad a = b\frac{\sin\alpha}{\sin\beta} = \frac{421 \times \sin 35^\circ 17'}{\sin 45^\circ 13'} = \frac{421(0.5776)}{0.7098}$$

$\therefore$ a = 342.58 = 343 approximately.

Again $$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$

$$\therefore \quad c = b\frac{\sin\gamma}{\sin\beta} = \frac{421 \times \sin 99^\circ 30'}{\sin 45^\circ 13'} = \frac{421(0.9863)}{0.7098}$$

= 584.99 = 585 approximately.

Hence $\gamma = 99^\circ 30'$, $a = 343$, $c = 585$.

Exercise 12.4

Solve the triangle *ABC*, if

1. $\beta = 60^\circ$ = , γ 15° , b $\sqrt{6}$

2. $\beta = 52^\circ$ = , γ $89^\circ 35'$, a 89.35

3. $b = 125 \quad =, \qquad \gamma \quad 53^\circ \qquad , \qquad \alpha \quad 47^\circ$

4. $c = 16.1 = \qquad , \qquad \alpha \quad 42^\circ 45' \qquad , \qquad \gamma \quad 74^\circ 32'$

5. $a = 53 \quad =, \qquad \beta \quad 88^\circ 36' \qquad , \qquad \gamma \quad 31^\circ 54'$

12.7.2 Case II: When measures of two sides and their included angle are given

In this case, we can use any one of the following methods:

i) First law of cosine and then law of sines,

or ii) First law of tangents and then law of sines.

Example 1: Solve the triangle *ABC*, by using the cosine and sine laws, given that *b* = 3, *c* = 5 and *a* = 120°.

Solution: By cosine laws,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha = 9 + 25 - 2(3)(5)\cos 120^\circ$$

$$= 9 + 25 - 2(3)(5)\left(-\frac{1}{2}\right) = 9 + 25 + 15 = 49$$

$\therefore$ a = 7

NOW $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$$\Rightarrow \quad \sin\beta = \frac{b\sin\alpha}{a} = \frac{3\times\sin 120^\circ}{7} = \frac{3\times 0.866}{7} = 0.3712$$

$\therefore \quad \beta = 21^\circ 47'$

$\therefore \quad \gamma = 180^\circ - (\alpha + \beta) = 180^\circ - (120^\circ + 21^\circ 47')$

$\gamma = 88^\circ 13'$

Hence $a = 7, \beta = 21^\circ 47'$ and $\gamma = 38^\circ 13'$

Example 2: Solve the triangle ABC, in which:

$$a = 36.21, \; c = 30.14, \quad \beta = 78^\circ 10'$$

Solution: Here $a > c \quad \therefore \; \alpha > \gamma$

$\because \quad \alpha + \beta + \gamma = 180^\circ$

$\because \quad \alpha + \gamma = 180^\circ - \beta = 180^\circ - 78^\circ 10'$

$\Rightarrow \quad \alpha + \gamma = 101^\circ 50'$

$\Rightarrow \quad \dfrac{\alpha + \gamma}{2} = 50^\circ 55'$

$\therefore$ By the law of tangents,

$$\frac{\tan\frac{\alpha-\gamma}{2}}{\tan\frac{\alpha+\gamma}{2}} = \frac{a-c}{a+c} \Rightarrow \tan\frac{\alpha-\gamma}{2} = \frac{a-c}{a+c}\tan\frac{\alpha+\gamma}{2}$$

so $\quad \tan\dfrac{\alpha-\gamma}{2} = \dfrac{36.21 - 30.14}{36.21 + 30.14} \cdot \tan 50^\circ 55'$

$\tan\dfrac{\alpha-\gamma}{2} = \dfrac{6.07}{66.35} \times 1.2312$

$\Rightarrow \quad \tan\dfrac{\alpha-\gamma}{2} = 0.1126$

$\Rightarrow \quad \tan\dfrac{\alpha-\gamma}{2} = 6^\circ 26'$

$\alpha - \gamma = 12^\circ 52' \qquad$ (ii)

Solving (i) and (ii) we have

$\alpha = 57^\circ 21$ and $\gamma \quad 44^\circ 29'$

To find side *b*, we use law of sines

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \Rightarrow b = \frac{a\sin\beta}{\sin\alpha}$$

$$b = \frac{36.21\times\sin 78^\circ 10'}{\sin 57^\circ 21'} \quad \frac{(36.21)(0.9788)}{(0.8420)} \quad 420.09$$

Hence $b = 42.09, \; \gamma = 44^\circ 29'$ and $\alpha = 57^\circ 21'$

Example 3: Two forces of 20 Newtons and 15 Newtons, inclined at an angle of 45°, are applied at a point on a body. If these forces are represented by two adjacent sides of a parallelogram then, their resultant is represented by its diagonal. Find the resultan force and also the angle which the resultant makes with the force of 20 Newtons.

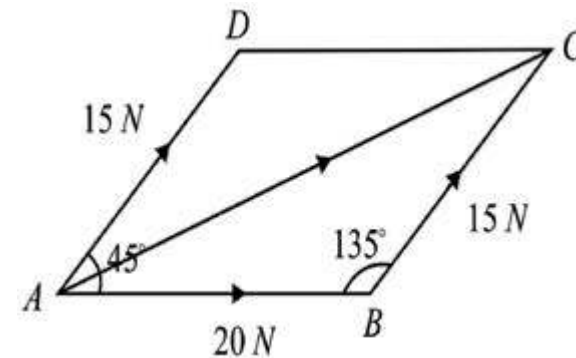
Solution:

Let *ABCD* be a llᵐ, such that

$|\overrightarrow{AB}|$ represent 20 Newtons

$|\overrightarrow{AD}|$ represents 15 Newtons

and $m\angle BAD = 45^\circ$

$\because$ $ABCD$ is a $\|^m$

$$\therefore \begin{cases} |\overrightarrow{BC}| = |\overrightarrow{AD}| = 15\ N \\ m\angle ABC = 180^\circ - m\angle BAD = 180^\circ - 45^\circ = 135^\circ \end{cases}$$

By the law of cosine,

$$\left(|\overrightarrow{AC}|\right)^2 = \left(|\overrightarrow{AB}|\right)^2 + \left(\overrightarrow{BC}\right)^2 - 2|\overrightarrow{AB}| \times |\overrightarrow{BC}| \times \cos 135^\circ$$

$$= (20)^2 + (15)^2 - 2\times 20\times 15\times\frac{-1}{\sqrt{2}}$$

$$= 400 + 225 + 424.2$$

$$= 1049.2$$

$$\therefore \quad |\overrightarrow{AC}| = \sqrt{1049.2} = 32.4\ N$$

By the law of sines,

$$\frac{|\overrightarrow{BC}|}{\sin m\angle BAC} = \frac{|\overrightarrow{AC}|}{\sin 135^\circ}$$

Make $|\overrightarrow{AB}|, |\overrightarrow{BC}|, |\overrightarrow{AD}|$ and $|\overrightarrow{AC}|$

$$\therefore \quad \sin m\angle BAC = \frac{|\overrightarrow{BC}| \times \sin 135^\circ}{|\overrightarrow{AC}|} = \frac{15\times 0.707}{32.4} = 0.3274$$

$$\therefore \quad m\angle BAC = 19^\circ 6'$$

Exercise 12.5

Solve the triangle *ABC* in which:

1. $b = 95$ $c = 34$ and $\alpha = 52^\circ$

2. $b = 12.5$ $c = 23$ and $\alpha = 38^\circ 20'$

3. $a = \sqrt{3} - 1$ $b = \sqrt{3} + 1$ and $\gamma = 60^\circ$

4. $a = 3$ $c = 6$ and $\beta = 36^\circ 20'$

5. $a = 7$ $b = 3$ and $\gamma = 38^\circ 13'$

Solve the following triangles, using first Law of tangents and then Law of sines:

6. $a == 36.21 = b$ 42.09 and γ $44^\circ 29'$

7. $a == 93 = b$ 101 and β 80°

8. $a == 14.8 = c$ 16.1 and α $42^\circ 45'$

9. $a == 319 = b$ 168 and γ $110^\circ 22$

10. $a == 61 = a$ 32 and α $59^\circ 30$

11. Measures of two sides of a triangle are in the ratio 3 : 2 and they include an angle of measure 57°. Find the remaining two angles.

12. Two forces of 40 *N* and 30 *N* are represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ which are inclined at an angle of 147° 25″. Find $\overrightarrow{AC}$, the resultant of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

12.7.3 Case. lll: When Measures of Three Sides are Given

In this case, we can take help of the following formulas:

i) the law of cosine;

or ii) the half angle formulas:

Example 1: Solve the triangle *ABC*, by using the law of cosine when
$a = 7, b = 3, c = 5$

Solution: We know that

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$\therefore \quad \cos\alpha = \frac{9+25-49}{30} = \frac{15}{30} \quad \frac{1}{2}$$

$$\therefore \quad \alpha = 120^\circ$$

$$\cos\beta = \frac{c^2+a^2-b^2}{2ca} = \frac{25+49-9}{70} = \frac{65}{70} \quad 0.9286$$

$$\therefore \quad \beta = 21^\circ 17'$$

and $\gamma = 180^\circ - (\alpha+\beta) = 180^\circ - (120^\circ + 21^\circ 47') = 38^\circ 13'$

Example 2: Solve the triangle *ABC*, by half angle formula, when
$a = 283, \quad b = 317, c = 428$

Solution: $2s = a + b + c = 283 + 317 + 428 = 1028$

$s = 514$

$s - a = 514 - 283 = 231$

$s - b = 514 - 317 = 197$

$s - c = 514 - 428 = 96$

Now, $$\tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad \sqrt{\frac{197\times 86}{514\times 231}} \quad 0.3777$$

$$\frac{\alpha}{2} = 20^\circ 53' \Rightarrow \alpha = 41^\circ 24'$$ and

$$\tan\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \quad \sqrt{\frac{86\times 231}{514\times 197}} \quad 0.4429$$

$$\therefore \quad \frac{\beta}{2} = 23^\circ 53' \Rightarrow \beta = 47^\circ 46'$$

$$\therefore \quad \gamma = 180^\circ - (\alpha+\beta) = 180^\circ - (41^\circ 24' + 47^\circ 46') = 90^\circ 50'$$

Exercise 12.6

Solve the following triangles, in which

1. $a = 7$, $b = 7$, $c = 9$
2. $a = 32$, $b = 40$, $c = 66$
3. $a = 28.3$, $b = 31.7$, $c = 42.8$
4. $a = 31.9$, $b = 56.31$, $c = 40.27$
5. $a = 4584$, $b = 5140$, $c = 3624$
6. Find the smallest angle of the triangle *ABC*, when $a = 37.34$, $b = 3.24$, $c = 35.06$.
7. Find the measure of the greatest angle, if sides of the triangle are 16, 20, 33.
8. The sides of a triangle are x^2+x+1, $2x+1$ and x^2-1. Prove that the greatest angle of the triangle is 120°.
9. The measures of side of a triangular plot are 413, 214 and 375 meters. Find the measures of the comer angles of the plot.
10. Three villages *A*, *B* and *C* are connected by straight roads 6 km. 9 km and 13 km. What angles these roads make with each other?

12.8 Area of Triangle

We have learnt the methods of solving different types of triangle. Now we shall find the methods of finding the area of these triangles.

case 1 Area of Triangle in Terms of the Measures of Two Sides and Their Included Angle

With usual notations, prove that:

$$\text{Area of triangle } ABC = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta = \frac{1}{2}ab\sin\gamma$$

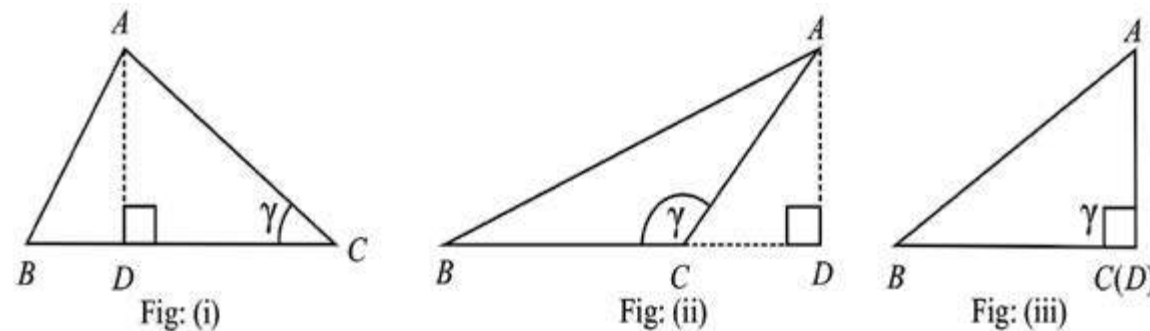
Proof: Consider three different kinds of triangle *ABC* with $m\angle C = \gamma$ as

i) acute ii) obtuse and iii) right

From *A*, draw $\overline{AD} \perp \overline{BC}$ or $\overline{BC}$ produced.

Fig: (i)

Fig: (ii)

Fig: (iii)

In figure. (i), $\frac{\overline{AD}}{\overline{AC}} = \sin\gamma$

In figure. (ii), $\frac{\overline{AD}}{\overline{AC}} = \sin(180^\circ - \gamma) = \sin\gamma$

In figure. (iii), $\frac{\overline{AD}}{\overline{AC}} = 1 = \sin 90^\circ = \sin\gamma$

In all the three cases, we have

$$\overline{AD} = \overline{AC}\sin\gamma = b\sin\gamma$$

Let Δ denote the area of triangle *ABC*.

By elementary geometry we know that

$$\Delta = \frac{1}{2}\text{(base)(altitude)}$$

$$\therefore \quad \Delta = \frac{1}{2}\overline{BC}\,.\,\overline{AD}$$

$$\therefore \quad \Delta = \frac{1}{2}ab\sin\gamma$$

Similarly, we can prove that:

$$\Delta = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin$$

Case II. Area of Triangle in Terms of the Measures of One Side and two Angles

In a triangle ΔABC, with usual notations, prove that:

$$\text{Area of triangle} = \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha} = \frac{b^2\sin\gamma\sin\alpha}{2\sin\beta} = \frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$$

Proof: By the law of sines, we know that:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\Rightarrow \quad a = c\frac{\sin\alpha}{\sin\gamma} \quad \text{and} \quad b = c\frac{\sin\beta}{\sin\gamma}$$

We know that area of triangle *ABC* is

$$\Delta = \frac{1}{2}ab\sin\gamma$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\left(c\frac{\sin\alpha}{\sin\gamma}\right)\left(\frac{c\sin\beta}{\sin\gamma}\right)\sin\gamma$$

$$\therefore \quad \Delta = \frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$$

In a similar way, we can prove that:

$$\Delta ABC = \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha} \quad \text{and} \quad \Delta ABC = \frac{b^2\sin\gamma\sin\alpha}{2\sin\beta}$$

Case III. Area of Triangle in Terms of the Measures of its Sides

In a triangle *ABC*, with usual notation, prove that:

$$\text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c}$$

Proof: We know that area of triangle *ABC* is

$$\Delta = \frac{1}{2}bc\sin\alpha$$

$$= \frac{1}{2} bc \,.\, 2 \sin\frac{\alpha}{2} \cos\frac{\alpha}{2} \qquad \left(\sin\alpha \quad 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \right)$$

$$= bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}} \text{ (by half angle formulas)}$$

$$= bc \frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc}$$

$$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Which is also called **Hero's formula**

Example 1: Find the area of the triangle ABC, in which
b = 21.6, c = 30.2 and a = 52° 40′

Solution: We know that:

$$\Delta ABC = \frac{1}{2} bc \sin\alpha = \frac{1}{2}(21.6)(30.2) \sin 52^\circ\, 40'$$

$$= \frac{1}{2}(21.6)(30.2)(0.7951)$$

$$\therefore \quad \Delta ABC = 259.3 \text{ sq.units}$$

Example 2: Find the area of the triangle *ABC*, when

$\alpha = 35^\circ\, 17'$, γ $45^\circ 13'$ and b 42.1

Solution: $\because \alpha + \beta + \gamma = 180^\circ$

$$\therefore \quad \beta = 180^\circ - (\alpha + \gamma) = 180^\circ - (35^\circ 17' + 45^\circ 13') = 99^\circ\, 30'$$

Also $b = 42.1$ α $35^\circ 17'$, γ $45^\circ 13'$, β $99^\circ\, 30'$

We know that the area of triangle *ABC* is

$$\Delta = \frac{1}{2} \frac{b^2 \sin\gamma \sin\alpha}{\sin\beta}$$

$$\therefore \quad = \frac{1}{2} \frac{(42.1)^2 \sin 45^\circ 13' \sin 35^\circ\, 17'}{\sin 99^\circ\, 30'}$$

$$= \frac{1}{2} \frac{(42.1)^2 (0.7097)(0.5776)}{(0.9863)}$$

$\therefore \quad \Delta = 368.3$ square units.

Example 3: Find the area of the triangle *ABC* in which
a = 275.4, b = 303.7, c = 342.5

Solution: $\therefore \quad a = 275.4, b = 303.7, \ c = 342.5$

∴ 2s = a + b + c
= 275.4 + 303.7 + 342.5 = 921.6

∴ s = 460.8

Now $s - a$ = 460.8 – 275.4 =185.4
$s - b$ = 460.8 – 303.7=157.1
$s - c$ = 460.8 – 342.5 =118.3

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{460.8 \times 185.4 \times 157.1 \times 118.3}$$

$$\therefore \quad \Delta = 39847 \text{ sq. units}$$

Exercise 12.7

1. Find the area of the triangle *ABC*, given two sides and their included angle:

i) $a = 200$, b 120, γ 150°

ii) $b = 37$, c 45, α $30^\circ\, 50'$

iii) $b = 4.33$, b 9.25, γ $56^\circ\, 44'$

2. Find the area of the triangle *ABC*, given one side and two angles:

i) $b = 25.4$, γ $36^\circ\, 41'$, α $45^\circ 17'$

ii) $c = 32$, α $47^\circ\, 24'$, β $70^\circ 16'$

iii) $a = 8.2$, α $83^\circ\, 42'$, γ $37^\circ 12'$

3. Find the area of the triangle *ABC*, given three sides:

i) $a = 18$, $b = 24$, $c = 30$

ii) $a = 524$, $b = 276$, $c = 315$

iii) $a = 32.65$, $b = 42.81$, $c = 64.92$

4. The area of triangle is 2437. If $a = 79$, and $c = 97$, then find angle β.

5. The area of triangle is 121.34. If $\alpha = 32° 15$ $\beta = 65° 37$ then find *c* and angle γ.

6. One side of a triangular garden is 30 m. If its two corner angles are $22^{\circ}\frac{1}{2}$ and $112^{\circ}\frac{1}{2}$, find the cost of planting the grass at the rate of Rs. 5 per square meter.

12.9 Circles Connected with Triangle

In our previous classes, we have learnt the methods of drawing the following three kinds of circles related to a triangle:

i) Circum-Circle ii) In-Circle iii) Ex-Circle.

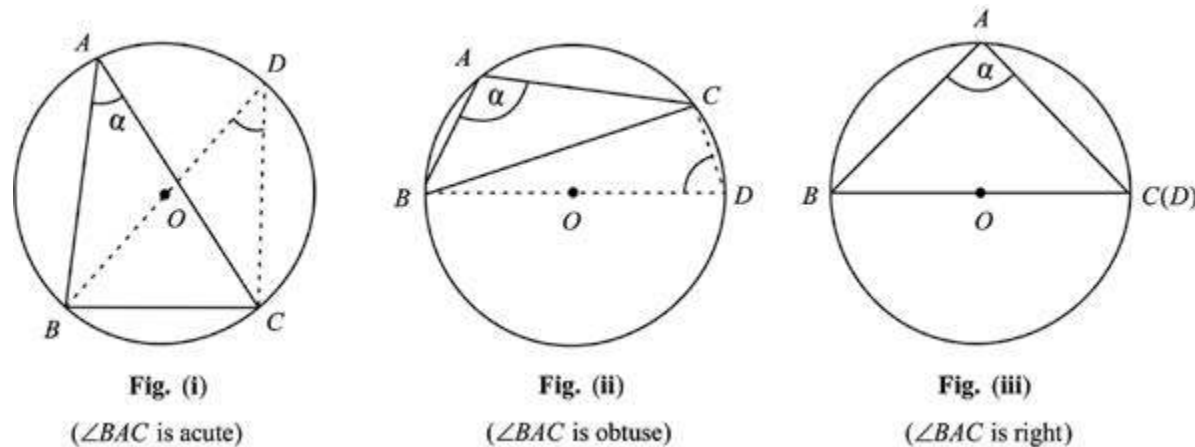
12.9.1 Circum-Circle:

The circle passing through the three vertices of a triangle is called a **Circum- Circle**. Its centre is called the **circum-centre**, which is the point of intersection of the right bisectors of the sides of the triangle. Its radius is called the **circum-radius** and is denoted by *R*.

a) Prove that: $R = \dfrac{a}{2\sin\alpha} = \dfrac{b}{2\sin\beta} = \dfrac{c}{2\sin\gamma}$ with usual notations.

Fig. (i)
($\angle BAC$ is acute)

Fig. (ii)
($\angle BAC$ is obtuse)

Fig. (iii)
($\angle BAC$ is right)

Proof: Consider three different kinds of triangle *ABC* with $m\angle A = \alpha$

i) acute ii) obtuse iii) right.

Let *O* be the circum-centre of ΔABC. Join *B* to *O* and produce $\overline{BO}$ to -meet the circle again at *D*. Join *C* to *D*. Thus we have the measure of diameter $m\overline{BD} == 2R$ and $m\overline{BC}$ a

I. In fig. (i), $m\angle BDC = m\angle A = \alpha$ (Angles in the same segment)

In right triangle *BCD*,

$$\frac{m\ \overline{BC}}{m\ \overline{BD}} = \sin m\angle BDC = \sin\alpha$$

II. In fig. (ii),

$$m\angle BDC + m\angle A = 180^{\circ}$$ (Sum of opposite angles of a cyclic quadrilateral 180°)

$$\Rightarrow \quad m\angle BDC + \alpha = 180^{\circ}$$

$$\Rightarrow \quad m\angle BDC = 180^{\circ} - \alpha$$

In right triangle *BCD*,

$$\frac{m\ \overline{BC}}{m\ \overline{BD}} = \sin m\angle BDC = \sin(180^{\circ} - \alpha) = \sin\alpha$$

III. In fig. (iii), $m\angle A = \alpha = 90^{\circ}$

$$\therefore \quad \frac{m\ \overline{BC}}{m\ \overline{BD}} = 1 \quad \sin 90^{\circ} \quad \sin\alpha$$

In all the three figures, we have proved that

$$\frac{m\ \overline{BC}}{m\ \overline{BD}} = \sin\alpha$$

$$\Rightarrow \quad \frac{a}{2R} = \sin\alpha \Rightarrow 2R\sin\alpha = a$$

$$\therefore \quad R = \frac{a}{2\sin\alpha}$$

Similarly, we can prove that

$$R = \frac{b}{2\sin\beta} \quad \text{and} \quad R = \frac{R.c}{2\sin\gamma}$$

Hence $\quad R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$

a) Deduction of Law of Sines:

We know that $R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$

$$\Rightarrow \quad \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

$\therefore \quad \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$, which is the law of sines.

b) Prove that: $R = \frac{abc}{4\Delta}$

Proof: We know that: $R = \frac{a}{2\sin\alpha}$

$$\Rightarrow \quad R = \frac{a}{2.2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} \qquad \left(\because \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$$

$$= \frac{a}{4\sqrt{\frac{s(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}} \quad \text{(by half angle formulas)}$$

$$= \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

$$\therefore \quad R = \frac{abc}{4\Delta} \qquad \left(\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}\right)$$

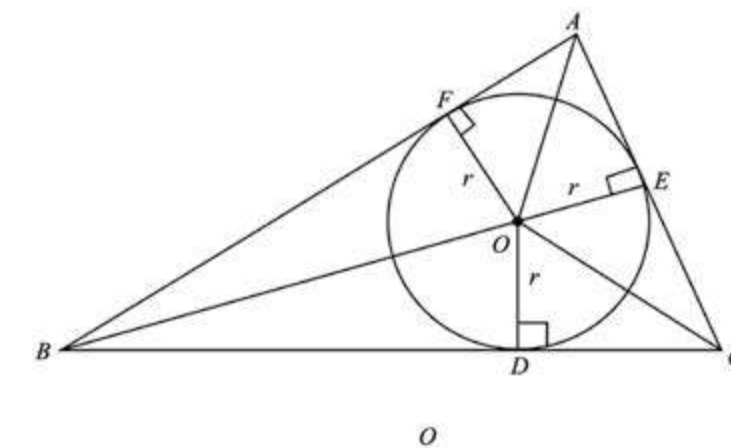
12.9.2 In-Circle

The circle drawn inside a triangle touching its three sides is called its **inscribed circle** or **in-circle**. Its centre is known as the **in-centre**, it is the point of intersection of the bisectors of angles of the triangle. Its radius is called **in-radius** and is denoted by *r*.

a) Prove that: $r = \frac{\Delta}{s}$ with usual notations.

Proof: Let the internal bisectors of angles of triangle *ABC* meet at *O*, the in-centre

Draw $\overline{OD} \perp \overline{BC}$, $\overline{OE} \perp \overline{AC}$ and $\overline{OF} \perp \overline{AB}$

O

Let, $m\overline{OD} = m\overline{OE} = m\overline{OF} = r$

From the figure Area $\Delta ABC =$ Area$\Delta OBC +$ Area$\Delta OCA +$ AreaΔOAB

$$\therefore \quad \Delta = \frac{1}{2}\overline{BC}\times\overline{OD} + \frac{1}{2}\overline{CA}\times\overline{OE} + \frac{1}{2}\overline{AB}\times\overline{OF}$$

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

$$= \frac{1}{2}r(a+b+c)$$

$$\Delta = \frac{1}{2}r.2s \qquad (\because 2s = a+b+c)$$

$$\Rightarrow \boxed{r = \frac{\Delta}{s}}$$

12.9.3 Escribed Circles

A circle, which touches one side of the triangle externally and the other two produced sides, is called an **escribed circle** or **ex-circle** or **e-circle**. Obviously, there could be only three such circles of a triangle, one opposite to each angle of the triangle.

The centres of these circles, which are called **ex-centres** are the points where the internal bisector of one and the external bisectors of the other two angles of the triangle meet.

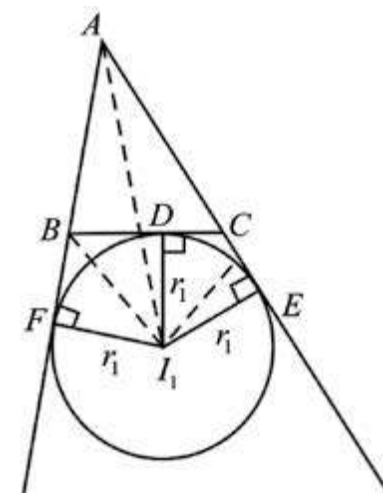
In ΔABC, centre of the ex-circle opposite to the **vertex A** is usually taken as I_1 and its raidus is denoted by r_1. Similarly, centres of ex-circles opposite to the vertices B and C are taken as I_2 and I_3 and their radii are denoted by r_2 and r_3 respectively.

a) With usual notation, prove that:

$$r_1 = \frac{\Delta}{s-a},\quad r_2 = \frac{\Delta}{s-b},\quad \text{and } r_3 = \frac{\Delta}{s-c}$$

Proof: Let I_1 be the centre of the escribed circle opposite to the vertex A of ΔABC,

From I_1 draw $\overline{I_1D} \perp \overline{BC}$, $\overline{I_1E} \perp \overline{AC}$ produced and $\overline{I_1F} \perp \overline{AB}$ produced.

Join I_1 to A, B and C.

Let $m\overline{I_1D} = m\overline{I_1E} = m\overline{I_1F} = r_1$

From the figure

$$\Delta ABC = \Delta I_1AB + \Delta I_1AC - \Delta I_1BC$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\overline{AB} \times \overline{I_1F} + \frac{1}{2}\overline{AC} \times \overline{I_1E} - \frac{1}{2}\overline{BC} \times \overline{I_1D}$$

$$= \frac{1}{2}c\,r_1 + \frac{1}{2}b\,r_1 - \frac{1}{2}a\,r_1$$

$$\Delta = \frac{1}{2}r_1(c+b-a)$$

$$= \frac{1}{2}r_1 \cdot 2(s-a) \qquad (2s = a+b+c)$$

$$= (s-a)\, r_1$$

Hence $\boxed{r_1 = \dfrac{\Delta}{s-a}}$

In a similar way, we can prove that:

$$\boxed{r_2 = \frac{\Delta}{s-b}} \quad \text{and} \quad \boxed{r_3 = \frac{\Delta}{s-c}}$$

Example 1: Show that:

$$r = (s-a)\tan\frac{\alpha}{2} = (s-b)\tan\frac{\beta}{2} = (s-c)\tan\frac{\gamma}{2}$$

Solution: To prove $r = (s-a)\tan\dfrac{\alpha}{2}$

We know that: $\tan\dfrac{\alpha}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

$$\text{R.H.S} = (s-a)\tan\frac{\alpha}{2} = (s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}} = \frac{\Delta}{s} = r$$

$$\therefore \quad (s-a)\tan\frac{\alpha}{2} = r$$

In a similar way, we can prove that:

$$r = (s-b)\tan\frac{\beta}{2} \quad \text{and} \quad r = (s-c)\tan\frac{\gamma}{2}$$

Example 2: Show that $r_1 = 4R\sin\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}$.

Solution: R.H.S. $= 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$

$$= 4 \cdot \frac{abc}{4\Delta}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{s(s-b)(s-c)}{\Delta}$$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta \cdot (s-a)}$$

$$= \frac{\Delta^2}{\Delta(s-a)}$$

$$= \frac{\Delta}{s-a} = r_1 = \text{L.H.S}$$

Hence $r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$

Example 3 : Prove that $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$

Solution: L.H.S. $= \frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}$

$$= \frac{s^2}{\Delta^2}+\frac{(s-a)^2}{\Delta^2}+\frac{(s-b)^2}{\Delta^2}+\frac{(s-c)^2}{\Delta^2}$$

$$= \frac{s^2+(s-a)^2+(s-b)^2+(s-c)^2}{\Delta^2}$$

$$= \frac{4s^2-2s(a+b+c)+a^2+b^2+c^2}{\Delta^2}$$

$$= \frac{4s^2-2s.2s+a^2+b^2+c^2}{\Delta^2}$$

$$= \frac{a^2+b^2+c^2}{\Delta^2}$$

= R.H.S.

Hence the result.

Example 4: If the measures of the sides of a triangle *ABC* are 17, 10, 21. Find R, r, r_1, r_2 and r_3.

Solution: Let $a = 17, \quad b = 10, \quad c = 21$

$\therefore \quad 2s = a+b+c = 17+10+21 = 48$

$\Rightarrow \quad S = 24$

$\therefore \quad s-a = 24-17 = 7,\ s-b = 24-10 = 14$ and $s-c = 24-21 = 3$

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Rightarrow \quad \Delta = \sqrt{24(7)(14)(3)} \quad 84$

Now $R == \frac{abc}{4\Delta} = \frac{17 \cdot 10 \cdot 21}{4 \cdot 84} \quad \frac{85}{8}$

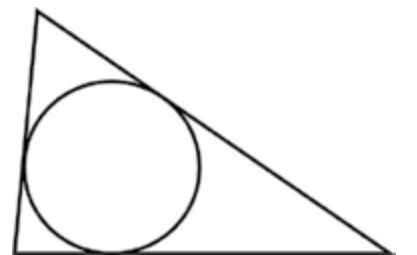
$r = \frac{\Delta}{s} = \frac{84}{24} = \frac{7}{2},\ r_1 = \frac{\Delta}{s-a} = \frac{84}{7} \quad 12,$

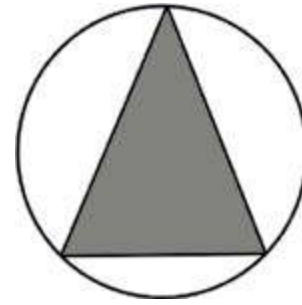
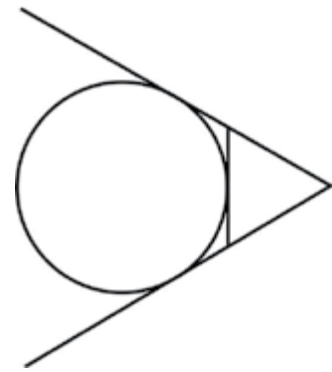
$r_2 = \frac{\Delta}{s-b} = \frac{84}{14} = 6,\ r_3 = \frac{\Delta}{s-c} = \frac{84}{3} \quad 28$

12.10 Engineering and Circles Connected With Triangles

We know that frames of all rectilinear shapes with the exception of triangular ones, change their shapes when pressed from two corners. But a triangular frame does not change its shape, when it is pressed from any two vertices. It means that a triangle is the only **rigid rectilinear figure.** It is on this account that the engineers make frequent use of triangles for the strength of material in all sorts of construction work.

Besides triangular frames etc., circular rings can stand greater pressure when pressed from any two points on them. That is why the wells are always made cylindrical whose circular surfaces can stand the pressure of water from all around their bottoms. Moreover, the arches below the bridges are constructed in the shape of arcs of circles so that they can bear the burden of the traffic passing over the bridge.

a) We know that triangular frames change their rectilinear nature when they are pressed from the sides. From the strength of material point of view, the engineers have to fix circular rings touching the sides of the triangular frames.

For making these rings, they have to find the **in-radii** of the triangles.

b) In order to protect the triangular discs from any kind of damage, the engineers fit circular rings enclosing the discs. For making rings of proper size, the engineers are bound to calculate the **circum-radii** of the triangles.

c) In certain triangular frames, the engineers have to extend two sides of the frames. In order to strengthen these loose wings, the engineer feels the necessity of fixing circular rings touching the extended sides andthe third side of the frames.

For making appropriate rings, the engineers have to find **ex-radii** of the triangles. The above discussion shows that the methods of calculations of the radii of **incircle, circum-circle and ex-circles** of traingles must be known to an engineer for performing his professional duty efficiently.

Exercise 12.8

1. Show that: $r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

ii) $s = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

2. Show that: $r == a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}$ $\quad b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2}$

$= c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$

3. Show that: i) $r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

ii) $r_2 = 4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$

iii) $r_3 = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$

4. Show that:

i) $r_1 = s\tan\frac{\alpha}{2}$ ii) $r_2 = s\tan\frac{\beta}{2}$ iii) $r_3 = s\tan\frac{\gamma}{2}$

5. Prove that:

i) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ ii) $rr_1\,r_2\,r_3 = \Delta^2$

iii) $r_1 + r_2 + r_3 - r = 4R$ iv) $r_1\,r_2\,r_3 = rs^2$

6. Find R, r, r_1 r_2 and r_3, if measures of the sides of triangle *ABC* are

i) $a = 13$, $b = 14$, $c = 15$

ii) $a = 34$, $b = 20$, $c = 42$

7. Prove that in an equilateral triangle,

i) $r : R : r_1 = 1 : 2 : 3$

ii) $r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$

8. Prove that:

i) $\Delta = r^2\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

ii) $r = s\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

iii) $\Delta = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

9. Show that: i) $\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

ii) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

10. Prove that:

$$r = \frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} = \frac{b\sin\frac{\alpha}{2}.\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}} = \frac{c\sin\frac{\alpha}{2}.\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

11. Prove that: $abc\,(\sin\alpha + \sin\beta + \sin\gamma) = 4\Delta s$

12. Prove that: i) $(r_1 + r_2)\tan\frac{\gamma}{2} = c.$

ii) $(r_3 - r)\cot\frac{\gamma}{2} = c$

CHAPTER

13

Invnerse Trignometric Functions

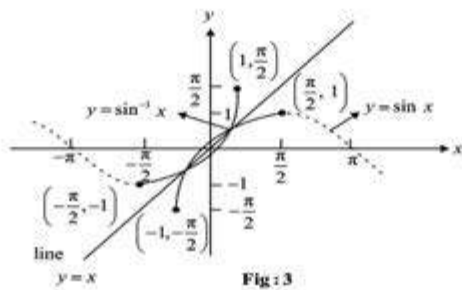
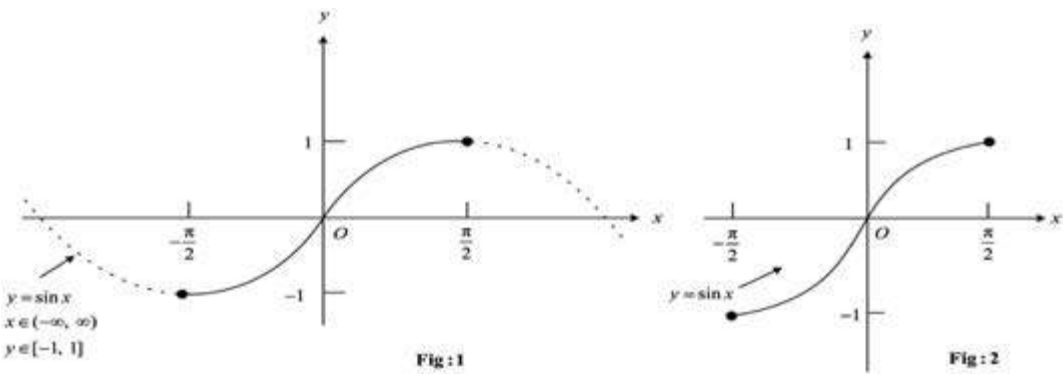
13.1 Introduction

We have been finding the values of trigonometric functions for given measures of the angles. But in the application of trigonometry, the problem has also been the other way round and we are required to find the measure of the angle when the value of its trigonometric function is given. For this purpose, we need to have the knowledge of **inverse trigonometric functions.**

In chapter 2, we have discussed inverse functions. We learned that only a one-to-one function will have an inverse. If a function is not one-to-one, it may be possible to restrict its domain to make it one-to-one so that its inverse can be found.

In this section we shall define the **inverse trigonometric functions.**

13.2 The Inverse sine Function:

The graph of $y = \sin x$, $-\infty < x < +\infty$, is shown in the figure 1.

We observe that every horizontal line between the lines y = 1 and y= -1 intersects the graph infinitly many times. It follows that the sine function is not one-to-one.However, if we

restrict the domain of $y = \sin x$ to the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then the restricted function $y = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is called the **principal sine function;** which is now one-to-one and hence will have an inverse as shown in figure 2.

This inverse function is called the **inverse sin function** and is written as $\sin^{-1}x$ or arc $\sin x$.

The **Inverse sine Function** is defined by:

$y = \sin^{-1}x$, if and only if $x = \sin y$.

where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $-1 \le x \le 1$

Here y is the angle whose sine is x. The domain of the function

$y = \sin^{-1}x$ is $-1 \le x \le 1$, its range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

The graph of $y = \sin^{-1}x$ is obtained by reflecting the restricted portion of the graph of $y = \sin x$ about the line $y = x$ as shown in figure 3.

We notice that the graph of $y = \sin x$ is along the x - axis whereas the graph of $y = \sin^{-1}x$ is along the y - axis.

Note: It must be remembered that $\sin^{-1}x \ne (\sin x)^{-1}$.

Example 1: Find the value of (i) $\sin^{-1}\frac{\sqrt{3}}{2}$ (ii) $\sin^{-1}(-\frac{1}{2})$

Solution: (i) We want to find the angle y, whose sine is $\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \sin y = \frac{\sqrt{3}}{2}, \quad \le \quad \le \quad \frac{\pi}{2} \quad y \quad \frac{\pi}{2}$

$\Rightarrow \quad y = \frac{\pi}{3}$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$$

(ii) We want to find the angle y whose sine is $-\frac{1}{2}$

$$\Rightarrow \quad \sin y=-\frac{1}{2}, \qquad -\frac{\pi}{2}\le y\le\frac{\pi}{2}$$

$$\therefore \quad -y=\frac{\pi}{6}$$

$$\therefore \quad \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$$

13.3 The Inverse Cosine Function:

The graph of y = cos$x$, $-\infty< x < +\infty$, is shown in the figure 4.

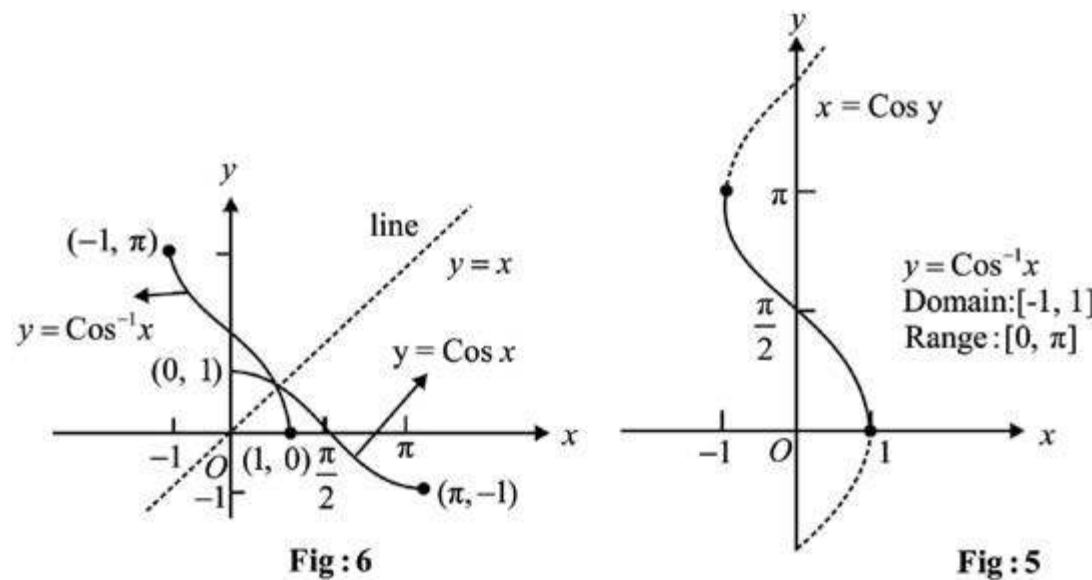
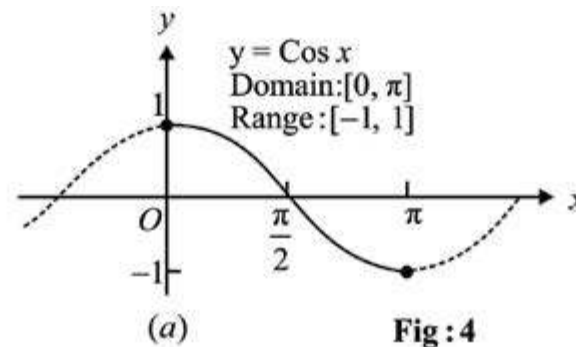



Fig : 4

Fig : 6 Fig : 5

We observe that every horizontal line between the lines y = 1 and y = -1 intersects the graph infinitly many times. It follows that the cosine function is not one-to-one. However, if we restrict the domain of y = cos$x$ to the interval [0, π], then the restricted function y = cos$x$, $0 \le x \le \pi$ is called the **principal cosine function;** which is now one-to-one and hence will have an inverse as shown in figure 5.

This inverse function is called the inverse cosine function and is written as $\cos^{-1}x$ or arc cos$x$.

The **Inverse Cosine Function** is defined by:

y = $\cos^{-1}x$, if and only if x= cos y.

where $0 \le y \le \pi$ and $-1 \le x \le 1$.

Here y is the angle whose cosine is x . The domain of the function y = $\cos^{-1}x$ is $-1 \le x \le 1$ and its range is $0 \le y \le \pi$.

The graph of y = $\cos^{-1}x$ is obtained by reflecting the restricted portion of the graph of y = cos x about the line y = x as shown in figure 6.

We notice that the graph of y = cos x is along the x - *axis* whereas the graph of y = $\cos^{-1}x$ is along the y - *axis* .

Note: It must be remembered that $\cos^{-1}x \ne (\cos x)^{-1}$

Example 2: Find the value of (i) $\cos^{-1}1$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution: (i) We want to find the angle y whose cosine is 1

$$\Rightarrow \quad \cos y=1, \qquad 0\le y\le\pi$$

$$\Rightarrow \quad y=0$$

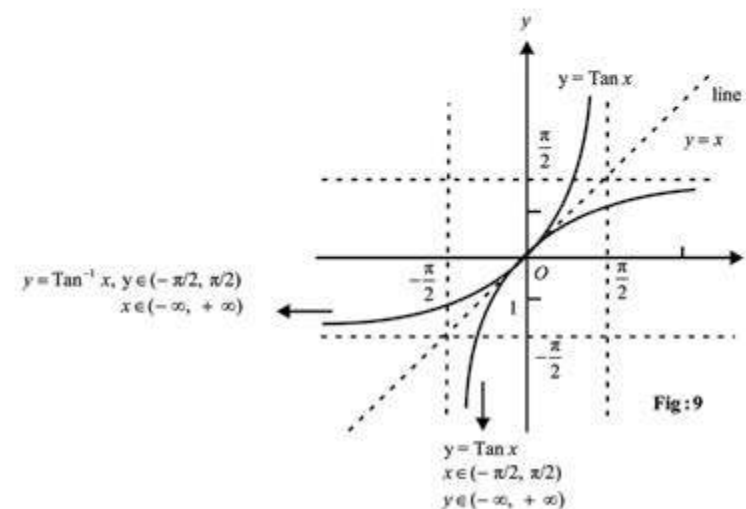
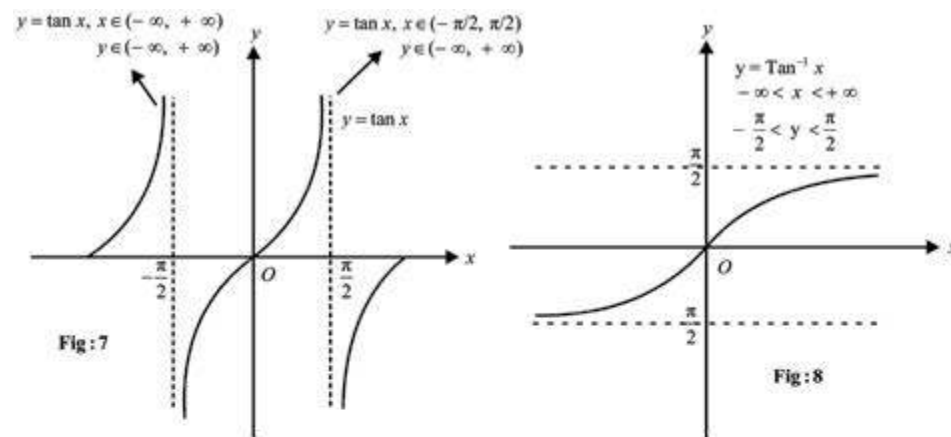
$$\therefore \quad \cos^{-1}1=0$$

(ii) We want to find the angle y whose cosine is $-\frac{1}{2}$

$$\Rightarrow \quad \cos y=-\frac{1}{2}, \quad 0\le y\le\pi$$

$\therefore \quad y = \frac{2\pi}{3}$

$\therefore \quad \cos^{-1}(-\frac{1}{2}) = -\frac{2\pi}{3}$

13.4 Inverse Tangent Function:

The graph of $y = \tan x$, $-\infty < x < +\infty$, is shown in the figure 7.
We observe that every horizontal line between the lines y = 1 and $y = -1$ intersect the graph infinitly many times. It follows that the tangent function is not one-to-one.

However, if we restrict the domain of y = *Tan*x to the interval $\frac{-\pi}{2} < x < \frac{\pi}{2}$, then the restricted

function y = tan$x$ $\frac{-\pi}{2} < x < \frac{\pi}{2}$, is called the **Principal tangent function;** which is now one-to-one and hence will have an inverse as shown in figure 8.

This inverse function is called the **inverse tangent function** and is written as $\tan^{-1}x$ or arc tan$x$.

The **Inverse Tangent Function** is defined by:

y = $\tan^{-1}x$, if and only if x = tan y.

where $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $-\infty < x < +\infty$.

Here y is the angle whose tangent is x. The domain of the function y = $\tan^{-1}x$ is $-\infty < x < +\infty$ and its range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$

The graph of y = $\tan^{-1}x$ is obtained by reflecting the restricted portion of the graph of y = tan$x$ about the line y = x as shown in figure 9.

We notice that the graph of y = tan$x$ is along the x - axis whereas the graph of y = tan$x$ is along the y- axis.

Note: It must be remembered that $\tan^{-1}x \neq (\tan x)^{-1}$.

Example 3: Find the value of (i) $\tan^{-1}1$ (ii) $\tan^{-1}(-\sqrt{3})$

Solution: (i) We want to find the angle y, whose tangent is 1

$\Rightarrow \quad \tan y = 1, \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow \quad y = \frac{\pi}{4}$

$\therefore \quad \tan^{-1} 1 = \frac{\pi}{4}$

(ii) We want to find the angle y whose tangent is $-\sqrt{3}$

$\Rightarrow \quad \tan y = -\sqrt{3} \qquad \frac{\pi}{2} \not< y \not< \frac{\pi}{2}$

$\therefore \quad y = \frac{2\pi}{3}$

$\therefore \quad \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$

13.5 Inverse Cotangent, Secant and Cosecant Functions

These inverse functions are not used frequently and most of the calculators do not even have keys for evaluating them. However, we list their definitions as below:

i) Inverse Cotangent function:

y = cot$x$, where $0 \le x \le \pi$ is called the **Principal Cotangent Function,** which is one-to-one and has an inverse.

The inverse cotangent function **is defined by:**

y = cot^{-1}x , if and only if x = coty

Where $0 < y < \pi$ and $-\infty < x < +\infty$

The students should draw the graph of y = cot^{-1} x by taking the reflection of y = cot$x$ in the line y = x. This is left as an exercise for them.

ii) Inverse Secant function

y = sec x, where $0 \le x \le \pi$ and $x \ne \frac{\pi}{2}$ is called the **Principal Secant Function,** which is one-to-one and has an inverse.

The Inverse Secant Function **is defined by:**

y = sec^{-1}x. if and only if x = secy

where $0 \le y \le \pi, \quad y \ne \frac{\pi}{2}$ and $|x| \ge 1$

The students should draw the graph of y = sec^{-1}x by taking the reflection of y = sec$x$ in the line y = x. This is left an exercise for them,

iii) Inverse Cosecant Function

y = csc x, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $x \ne 0$ is called the **Principal Cosecant Function,**

which is one-to-one and has an inverse.

The **Inverse Cosecant Function** is defined by:

$y = \csc^{-1} x$, if and only if $x = \csc y$

where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$ and $|x| \ge 1$

The students should draw the graph of $y = \csc^{-1}$ by taking the reflection of y = csc$x$ in the line y = x. This is left an exercise for them.

Note. While discusspjpihe Inverse Trigonometric Functions, we have seen that there are in general, no inverses of Trigonometric Functions, but restricting their domain to principal Functions, we have made them as functions.

13.6 Domains and Ranges of Principal Trigonometric Function and Inverse Trigonometric Functions.

From the discussion on the previous pages we get the following table showing domains and ranges of the Principal Trigonometric and Inverse Trigonometric Functions.

Functions	Domain	Range
$y = \sin x$	$\frac{-\pi}{2} \le x \le \frac{\pi}{2}$	$-1 \le x \le 1$
$y = \sin^{-1} x$	$-1 \le x \le 1$	$\frac{-\pi}{2} \le x \le \frac{\pi}{2}$
$y = \cos x$	$0 \le x \le \pi$	$-1 \le x \le 1$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le x \le \pi$
$y = \tan x$	$\frac{-\pi}{2} < x < \frac{\pi}{2}$	$(-\infty, \infty)$ or $\Re$

$y=\tan^{-1}x$	$(-\infty,\infty)$ or $\Re$	$\frac{-\pi}{2}<x<\frac{\pi}{2}$
$y=\cot x$	$0<x<\pi$	$(-\infty,\infty)$ or $\Re$
$y=\cot^{-1}x$	$(-\infty,\infty)$ or $\Re$	$0<x<\pi$
$y=\sec x$	$[0,\pi],\ x\neq\frac{\pi}{2}$	$y\leq -1$ or $y\geq 1$
$y=\sec^{-1}x$	$x\geq -1$ or $x\leq 1$	$[0,\pi],\ y\neq\frac{\pi}{2}$
$y=\csc x$	$[-\frac{\pi}{2},\frac{\pi}{2}],x\neq 0$	$y\leq -1$ or $y\geq 1$
$y=\csc^{-1}x$	$x\leq -1$ or $x\geq 1$	$[-\frac{\pi}{2},\frac{\pi}{2}],\ \ y\neq 0$

Example 4: Show that $\cos^{-1}\frac{12}{13} \quad \sin^{-1}\frac{5}{13}$

Solution: Let $\cos^{-1}\frac{12}{13} = \alpha \Rightarrow \cos\alpha = \frac{12}{13}$

$$\therefore \quad \sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\sqrt{1-\left(\frac{12}{13}\right)^2}$$

$$= \pm\sqrt{1-\frac{144}{169}}$$

$$= \pm\sqrt{\frac{169-144}{169}} = \pm\sqrt{\frac{25}{169}} = \frac{5}{13}$$

$\because$ cos α is +ve and domain of α is [0, π], in which sine is +ve.

Thus $\quad \sin\alpha = \frac{5}{13} \Rightarrow \quad \alpha = \sin^{-1}\frac{5}{13}$

Hence $\quad \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{5}{13}$

Example 5: Find the value of

i) $\sin(\cos^{-1}\frac{\sqrt{3}}{2})$ ii) $\cos(\tan^{-1}0)$ iii) $\sec[\sin^{-1}(-\frac{1}{2})]$ **Solution:**

i) we first find the value of y, whose cosine is $\frac{\sqrt{3}}{2}$

$$\cos y = \frac{\sqrt{3}}{2}, \quad 0\leq y\leq\pi$$

$$\Rightarrow \quad y = \frac{\pi}{6}$$

$$\Rightarrow \quad (\cos^{-1}\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$$

$$\therefore \quad \sin(\cos^{-1}\frac{\sqrt{3}}{2}) = \sin\frac{\pi}{6} \quad \frac{1}{2}$$

ii) we first find the value of y, whose tangent is 0

$$\tan y = 0, \quad -\frac{\pi}{2}<y<\frac{\pi}{2}$$

$$\Rightarrow \quad y = 0$$

$$\Rightarrow \quad (\tan^{-1}0) = 0$$

$$\therefore \quad \cos(\tan^{-1}0) = \cos 0 \quad 1$$

iii) we first find the value of y, whose sine is $-\frac{1}{2}$

$$\sin y = -\frac{1}{2}, \quad -\frac{\pi}{2}\leq y\leq\frac{\pi}{2}$$

$$\Rightarrow \quad -y = \frac{\pi}{6}$$

$$\Rightarrow \quad \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$$

$\therefore \quad \sec[\sin^{-1}(-\frac{1}{2})] = \frac{2}{\sqrt{3}}$

Example: 6 Prove that the inverse trigonometric functions satisfy the following identities:

i) $\sin^{-1}x = \frac{\pi}{2}\ \cos^{-1}x$ and $\cos^{-1}x\ \frac{\pi}{2}\ \sin^{-1}x$

ii) $\tan^{-1}x = \frac{\pi}{2}\ \cot^{-1}x$ and $\cot^{-1}x\ \frac{\pi}{2}\ \tan^{-1}x$

iii) $\sec^{-1}x = \frac{\pi}{2}\ \csc^{-1}x$ and $\csc^{-1}x\ \frac{\pi}{2}\ \sec^{-1}x$

Proof:

Consider the right triangle given in the figure Angles α and β are acute and complementary.

$\Rightarrow \quad \alpha + \beta = \frac{\pi}{2}$

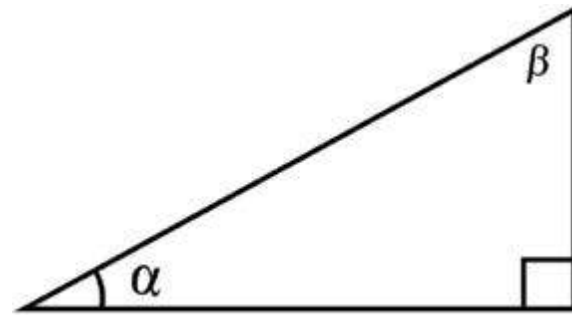
$\Rightarrow \quad \alpha = \frac{\pi}{2} - \beta$ and $\beta = \frac{\pi}{2} - \alpha \quad ...(i)$

Now $\sin\alpha = \sin(\frac{\pi}{2} - \beta) = \cos\beta = x$ (say)

$\therefore \quad \alpha = \sin^{-1}x$ and $\beta = \cos^{-1}x$

Thus from (i) we have:

$\sin^{-1}x = \frac{\pi}{2}\ \cos^{-1}x$ and $\cos^{-1}x\ \frac{\pi}{2}\ \sin^{-1}x$

In a similar way, we can derive the identities (ii) and (iii).

Exercise 13.1

1. Evaluate without using tables / calculator:

i) $\sin^{-1}(1)$ ii) $\sin^{-1}(-1)$ iii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

iv) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ v) $\cos^{-1}\left(\frac{1}{2}\right)$ vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

vii) $\cot^{-1}(-1)$ viii) $\operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ ix) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

2. Without using table/ Calculator show that:

i) $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$ ii) $2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$

iii) $\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$

3. Find the value of each expression:

i) $\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$ ii) $\sec\left(\cos^{-1}\frac{1}{2}\right)$ iii) $\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

iv) $\csc\left(\tan^{-1}(-1)\right)$ v) $\sec\left(\sin^{-1}(-\frac{1}{2})\right)$ vi) $\tan\left(\tan^{-1}(-1)\right)$

vii) $\sin\left(\sin^{-1}(\frac{1}{2})\right)$ viii) $\tan\left(\sin^{-1}(-\frac{1}{2})\right)$ ix) $\sin\left(\tan^{-1}(-1)\right)$

13.7 Addition and Subtraction Formulas

1) Prove that:

$\sin^{-1}A + \sin^{-1}B = \sin^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})$

Proof: Let $\sin^{-1}A = x \quad \Rightarrow \sin x = A$

and $\sin^{-1}B = y \quad \Rightarrow \sin y = B$

Now $\cos x = \pm\sqrt{1-\sin^2 x} = \pm\sqrt{1-A^2}$

In sin$x$ = A, domain = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, in which Cosine is +ve,

$\therefore \quad \cos x = \sqrt{1\quad A^2}$

Similarly, $\cos y = \sqrt{1-B^2}$

Now $\sin(x+y) = \sin x\cos y + \cos x\sin y$

$$= A\sqrt{1-B^2} + B\sqrt{1-A^2}$$

$$\Rightarrow \quad x+y = \sin^{-1}(A\sqrt{1-B^2} - B\sqrt{1-A^2}) \quad \therefore$$

$$\boxed{\sin^{-1}A + \sin^{-1}B = \sin^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})}$$

In a similar way, we can prove that

2) $\boxed{\sin^{-1}A - \sin^{-1}B = \sin^{-1}(A\sqrt{1-B^2} - B\sqrt{1-A^2})}$

3) $\boxed{\cos^{-1}A + \cos^{-1}B = \cos^{-1}(AB - \sqrt{(1-A^2)(1-B^2)})}$

4) $\boxed{\cos^{-1}A - \cos^{-1}B = \cos^{-1}(AB + \sqrt{(1-A^2)(1-B^2)})}$

5) Prove that:

$$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}$$

Proof: Let $\tan^{-1}A = x \Rightarrow \tan x = A$

and $\tan^{-1}B = y \Rightarrow \tan y = B$

Now $\tan(x+y) = \dfrac{\tan x + \tan y}{1-\tan x \tan y} = \dfrac{A+B}{1-AB}$

$$\Rightarrow \quad x+y = \tan^{-1}\frac{A+B}{1-AB}$$

$$\therefore \quad \boxed{\tan^{-1}A + \tan B = \tan^{-1}\frac{A+B}{1-AB}}$$

In a similar way, we can prove that

6) $\boxed{\tan^{-1}A - \tan^{-1}B = \tan^{-1}\dfrac{A-B}{1+AB}}$

Cor. Putting *A - B* in

$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\dfrac{A+B}{1-AB}$, we get

$$\tan^{-1}A + \tan^{-1}A = \tan^{-1}\frac{A+A}{1-A^2}$$

$$\Rightarrow 2\tan^{-1}A = \tan^{-1}\frac{2A}{1-A^2}$$

Exercise 13.2

Prove the following:

1. $\sin^{-1}\dfrac{5}{13} + \sin^{-1}\dfrac{7}{25} = \cos^{-1}\dfrac{253}{325}$

2. $\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{1}{5} = \tan^{-1}\dfrac{9}{19}$

3. $2\tan^{-1}\dfrac{2}{3} = \sin^{-1}\dfrac{12}{13}$ $\left[\textbf{Hint: } \text{Let } \tan^{-1}\dfrac{2}{3} = x \text{ and shown } \sin 2x = \dfrac{12}{13}\right]$

4. $\tan^{-1}\dfrac{120}{119} = 2\cos^{-1}\dfrac{12}{13}$

5. $\sin^{-1}\dfrac{1}{\sqrt{5}} + \cot^{-1}3 = \dfrac{\pi}{4}$

6. $\sin^{-1}\dfrac{3}{5} + \sin^{-1}\dfrac{8}{17} = \sin^{-1}\dfrac{77}{85}$

7. $\sin^{-1}\dfrac{77}{85} - \sin^{-1}\dfrac{3}{5} = \cos^{-1}\dfrac{15}{17}$

8. $\cos^{-1}\dfrac{63}{65} + 2\tan^{-1}\dfrac{1}{5} = \sin^{-1}\dfrac{3}{5}$

9. $\tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{3}{5} - \tan^{-1}\dfrac{8}{19} = \dfrac{\pi}{4}$

$\left[\textbf{Hint: } \text{First add } \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{3}{5} \text{ and then proceed}\right]$

10. $\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{5}{13} + \sin^{-1}\dfrac{16}{65} = \dfrac{\pi}{2}$

11. $\tan^{-1}\frac{1}{11} + \tan^{-1}\frac{5}{6} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}$

12. $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

13. Show that $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

14. Show that $\sin(2\cos^{-1} x) = 2x\sqrt{1-x^2}$

15. Show that $\cos(2\sin^{-1} x) = 1-2x^2$

16. Show that $\tan^{-1}(-x) - \quad = \quad \tan^{-1} x$

17. Show that $\sin^{-1}(-x) - \quad = \quad \sin^{-1} x$

18. Show that $\cos^{-1}(-x) \quad -= \pi \quad \cos^{-1} x$

19. Show that $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$

20. Given that $x = \sin^{-1}\frac{1}{2}$, find the values of following trigonometric functions: sin$x$, cos$x$, tan$x$, cot$x$, sec$x$ and csc$x$.

CHAPTER

14 Solutions of Trignometric Equation

14.1 Introduction

The Equations, containing at least one trigonometric function, are called **Trigonometric Equations**, *e.g.*, each of the following is a trigonometric equation:

$$\sin x = \frac{2}{5}, \ \text{Sec}\, x = \tan x \quad \text{and} \quad \sin^2 x \quad \sec x \quad 1 \quad \frac{3}{4}$$

Trigonometric equations have an infinite number of solutions due to the periodicity of the trigonometric functions. For example

$$\text{If} \ \sin\theta = \theta \ \text{then} \ \theta = 0, \pm \ , \pm 2 \ , \ldots$$

which can be written as $\theta =$, where $n \quad Z$.

In solving trigonometric equations, first find the solution over the interval whose length is equal to its period and then find the general solution as explained in the following examples:

Example 1: Solve the equation $\sin x = \frac{1}{2}$

Solution: $\sin x = \frac{1}{2}$

$\because$ sin x is positive in I and II Quadrants with the reference angle $x = \frac{}{6}$.

$$\therefore \ x = \frac{}{6} \ \text{and} \ x = \ - \frac{}{6} = \frac{5}{6} \ , \quad \text{where} \ x \in [0, 2 \ ]$$

$$\therefore \text{General values of } x \text{ are } \frac{}{6} + 2n \ \text{ and } \ \frac{5}{6} + 2n \ , \ n \in Z$$

Hence solution set $= \left\{\frac{}{6} + 2n \ \right\} \cup \left\{\frac{5}{6} + 2n \ \right\} \quad , n \in Z$

Example 2: Solve the equation: $1 + \cos x = 0$

Solution: $1 + \cos x = 0$

$\Rightarrow \quad \cos x = -1$

Since cos x is –ve, there is only one solution $x = \pi$ in $[0, 2\pi]$

Since 2π is the period of cos x

$\therefore$ General value of x is $\pi + 2n\pi$, $\quad n \in Z$

Hence solution set = $\{\pi + 2n\pi\}$, $\quad n \in Z$

Example 3: Solve the equation: $4\cos^2 x - 3 = 0$

Solution: $4\cos^2 x - 3 = 0$

$$\Rightarrow \cos^2 x = \frac{3}{4} \qquad \Rightarrow \pm \ \cos x = \frac{\sqrt{3}}{2}$$

i. If $\cos x = \frac{\sqrt{3}}{2}$

Since cos x is +ve in I and IV Quadrants with the reference angle

$$x = \frac{}{6}$$

$$\therefore \ x = \frac{}{6} \ \text{and} \ x = 2 \quad \frac{}{6} \quad \frac{11}{6} \qquad \text{where} \ x \in [0, 2 \ ]$$

As 2π is the period of cos x.

$$\therefore \ \text{General value of } x \text{ are } \frac{}{6} + 2n \ \text{ and } \ \frac{11}{6} + 2n \ , \quad n \in Z$$

ii. if $\cos x = -\frac{\sqrt{3}}{2}$

Since cos x is –ve in II and III Quadrants with reference angle $x = \frac{}{6}$

$$\therefore \ x = \ - \frac{}{6} = \frac{5}{6} \ \text{ and } \ x = x + \frac{}{6} = \frac{7}{6} \qquad \text{where} \ x \in [0, 2 \ ]$$

As 2π is the period of cos x.

$\therefore$ General values of x are $\frac{5}{6}+2n$ and $\frac{7}{6}+2n$, $n \in Z$

Hence solution set $=\left\{\frac{}{6}+2n\right\}\cup\left\{\frac{11}{6}+2n\right\}\cup\left\{\frac{5}{6}+2n\right\}\cup\left\{\frac{7}{6}+2n\right\}$

14.2 Solution of General Trigonometric Equations

When a trigonometric equation contains more than one trigonometric functions, trigonometric identities and algebraic formulae are used to transform such trigonometric equation to an equivalent equation that contains only one trigonometric function.

The method is illustrated in the following solved examples:

Example 1: Solve: sin x + cos x = 0.

Solution: sin x + cos x = 0

$\Rightarrow \frac{\sin x}{\cos x}+\frac{\cos x}{\cos x}=0$ (Dividing by $\cos x \neq 0$)

$\Rightarrow \tan x+1=0 \quad \Rightarrow \tan x = 1$

$\because$ tan x is –ve in II and IV Quadrants with the reference angle

$x=\frac{}{4}$

$\therefore \quad x = -\frac{}{4}=\frac{3}{4}$, where $x\in[0, \ ]$

As π is the period of tan x,

$\therefore$ General value of x is $\frac{3}{4}+n$, $n \in Z$

$\therefore$ Solution set = $\left\{\frac{3}{4}+n\right\}$, $n \in Z$.

Example 2: Find the solution set of: $\sin x\cos x=\frac{\sqrt{3}}{4}$.

Solution: $\sin x\cos x=\frac{\sqrt{3}}{4}$.

$\Rightarrow \frac{1}{2}(2\sin x\cos x)=\frac{\sqrt{3}}{4}$

$\Rightarrow \sin 2x=\frac{\sqrt{3}}{2}$

$\because$ sin $2x$ is +ve in I and II Quadrants with the reference angle $2x=\frac{}{3}$

$\therefore$ $2x=\frac{}{3}$ and $2x= -\frac{}{3}=\frac{2}{3}$ are two solutions in $[0,2\ ]$

As 2π is the period of sin $2x$.

$\therefore$ General values of $2x$ are $\frac{}{3}+2n$ and $\frac{2}{3}+2n$, , $n \in Z$

$\Rightarrow$ General values of x are $\frac{}{6}+n$ and $\frac{}{3}+n$, $n \in Z$

Hence solution set = $=\left\{\frac{}{6}+n\right\}\cup\left\{\frac{}{3}+n\right\}$, $n \in Z$

Note: In solving the equations of the form sin kx = c, we first find the solution pf sin u = c (where $kx = w$) and then required solution is obtained by dividing each term of this solution set by k.

Example 3: Solve the equation: sin $2x$ = cos $2x$

Solution: sin$2x$ = cos$2x$

$\Rightarrow$ 2sin$x$ cos x = cos$x$

$\Rightarrow$ 2sin$x$ cos x – cos$x$ = 0

$\Rightarrow$ cos$x$(2sin$x$ – 1) = 0

$\therefore$ cos$x$ = 0 or 2sin$x$ – 1 = 0

i. If cos$x$ = 0

$\Rightarrow$ $x = \frac{\;}{2}$ and $x = \frac{3}{2}$ where $x \in [0, 2\pi]$

As 2π is the period of cos x .

$\therefore$ General values of x are $\frac{\pi}{2}$ + $2n\pi$ and $\frac{3\pi}{2}$ + $2n\pi$, $n \in Z$,

ii. If 2 sin x – 1 = 0

$\Rightarrow$ sin x = $\frac{1}{2}$

Since sin x is +ve in I and II Quadrants with the reference angle $x = \frac{\pi}{6}$

$\therefore$ $x = \frac{\pi}{6}$ and $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ where $x \in [0, 2\pi]$

As 2π is the period of sin x.

$\therefore$ General values of x are and $\frac{\pi}{6}$ + $2n\pi$ and $5\frac{\pi}{6}$ + $2n\pi$, $n \in Z$,

Hence solution set = $\left[\frac{\pi}{2} + 2n\pi\right] \cup \left\{\frac{3\pi}{2} + 2n\pi\right\} \cup \left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{5\frac{\pi}{6} + 2n\pi\right\}$,

$n \in z$

Example 4: Solve the equation: $\sin^2 x + \cos x = 1$.

Solution: $\sin^2 x + \cos x = 1$

$\Rightarrow$ $1 - \cos^2 x + \cos x = 1$

$\Rightarrow$ $-\cos x(\cos x - 1) = 0$

$\Rightarrow$ $\cos x = 0$ or $\cos x - 1 = 0$

i. If cos x = 0

$\Rightarrow x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, where $x \in [0, 2\pi]$

As 2π is the period of cos x

$\therefore$ General values of x are $\frac{\pi}{2}$ + $2n\pi$ and $\frac{3\pi}{2}$ + $2n\pi$, $n \in Z$

ii. If cos x = 1

$\Rightarrow x = 0$ and $x = 2\pi$, where $x \in [0, 2\pi]$

As 2π is the period of cos x

$\therefore$ General values of x are $0 + 2n\pi$ and $2\pi + 2n\pi$, $n \in Z$.

$\therefore$ Solution Set = $\left\{\frac{\pi}{2} + 2n\pi\right\} \cup \left\{\frac{3\pi}{2} + 2n\pi\right\} \cup \{2n\pi\} \cup \{2\pi + 2n\pi\}, n \in z$

$\because \{2(n+1)\pi\} \subset \{2n\pi\},\ n \in z$

Hence the solution set = $\left[\frac{\pi}{2} + 2n\pi\right] \cup \left\{\frac{3\pi}{2} + 2n\pi\right\} \cup \{2n\pi\}, n \in z$

Sometimes it is necessary to square both sides of a trigonometric equation. In such a case, extaneous roots can occur which are to be discarded. So each value of x must be checked by substituting it in the given equation.

For example, $x = 2$ is an equation having a root 2. On squaring we get $x^2 - 4$ which gives two roots 2 and –2. But the root –2 does not satisfy the equation $x = 2$. Therefore, –2 is an extaneous root.

Example 5: Solve the equation: $\csc x = \sqrt{3} + \cot x$.

Solution: $\csc x = \sqrt{3} + \cot x$ ....(i)

$\Rightarrow \frac{1}{\sin x} = \sqrt{3} + \frac{\cos x}{\sin x}$

$\Rightarrow 1 = \sqrt{3}\sin x + \cos x$

$\Rightarrow 1 - \cos x = \sqrt{3}\sin x$

$\Rightarrow (1 - \cos x)^2 = (\sqrt{3}\sin x)^2$

$\Rightarrow 1-2\cos x+\cos^2 x = 3\sin^2 x$

$\Rightarrow 1-2\cos x+\cos^2 x = 3(1-\cos^2 x)$

$\Rightarrow 4\cos^2 x-2\cos x-2=0$

$\Rightarrow 2\cos^2 x-\cos x-1=0$

$\Rightarrow (2\cos x+1)(\cos x-1)=0$

$\Rightarrow \cos x = -\frac{1}{2}$ or $\cos x = 1$

i. If $\cos x = -\frac{1}{2}$

Since cos x is –v e in II and III Quadrants with the reference angle $x=\frac{\pi}{3}$

$\Rightarrow x=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$ and $x=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$, where $x\in[0, 2\pi]$

Now $x=\frac{4\pi}{3}$ does not satisfy the given equation (i).

$\therefore x=\frac{4\pi}{3}$ is not admissible and so $x=\frac{2\pi}{3}$ is the only solution.

Since 2π is the period of cos x

$\therefore$ General value of x is $\frac{2\pi}{3}+2n\pi$, $n\in Z$

ii. If cos x = 1

$\Rightarrow x = 0$ and $x = 2\pi$ where $x\in[0, 2\pi]$

Now both csc x and cot x are not defined for $x = 0$ and $x = 2$

$\therefore x = 0$ and $x = 2$ are not admissible.

Hence solution set = $\left\{\frac{2\pi}{3}+2n\pi\right\}$, $n\in Z$

Exercise 14

1. Find the solutions of the following equations which lie in $[0, 2\pi]$

i) $\sin x = -\frac{\sqrt{3}}{2}$ ii) $\text{cosec}\theta = 2$ iii) $\sec x = -2$ iv) $\cot\theta = \frac{1}{\sqrt{3}}$

2. Solve the following trigonometric equations:

i) $\tan^2\theta = \frac{1}{3}$ ii) $\text{cosec}^2\theta = \frac{4}{3}$ iii) $\sec^2\theta = \frac{4}{3}$ iv) $\cot^2\theta = \frac{1}{3}$

Find the values of θ satisfying the following equations:

3. $3\tan^2\theta + 2\sqrt{3}\tan\theta + 1 = 0$

4. $\tan^2\theta - \sec\theta - 1 = 0$

5. $2\sin\theta + \cos^2\theta - 1 = 0$

6. $2\sin^2\theta - \sin\theta = 0$

7. $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$ [Hint: Divide by $\sin^2\theta$]

Find the solution sets of the following equations.

8. $4\sin^2\theta - 8\cos\theta + 1 = 0$

9. $\sqrt{3}\tan x - \sec x - 1 = 0$

10. $\cos 2x = \sin 3x$ [Hint: $\sin 3x = 3\sin x - 4\sin^3 x$]

11. $\sec 3\theta = \sec\theta$

12. $\tan 2\theta + \cot\theta = 0$

13. $\sin 2x + \sin x = 0$

14. $\sin 4x - \sin 2x = \cos 3x$

15. $\sin x + \cos 3x = \cos 5x$

16. $\sin 3x + \sin 2x + \sin x = 0$

17. $\sin 7x - \sin x = \sin 3x$

18. $\sin x + \sin 3x + \sin 5x = 0$

19. $\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 0$

20. $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$